# Multiple Equations Model

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# **Covered Topics (1)**

Model Components

- Simultaneous Equations
  - -Endogenous variables
  - -Exogenous variables
- Cross-equation Correlation
- Cross-equation Restriction

# **Covered Topics (2)**

Estimation Methods for MEM

- Equation-by-equation estimation
- SURE
- Three-stage LS
- GMM

Test for Exogeneity ???

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# Simultaneous Equations Model (1)

Or SEM for short

Exogenous Variable

• its value is given

Endogenous Variable

• its value determined by the other variables (exog. or other endo.)

## Simultaneous Equations Model (2)

For an *M* equations model,

there can only be *M* endogenous variables. The other must be exogenous.

Random components or error terms must be exogenous, of course.

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## Simultaneous Equations Model (3)

Q: Who will say which variables in the model are endogenous and which are exogenous?

A: Generally, their ex/endogeneity is determined by the model structure. Easily speaking, they are assumed. However, they could be tested for exogeneity.

# **Structural Equations**

For a non-structural equation, variables can be on any side of the equations. But for a structural equation, the variable on the lefthand side is generally endogenous. Its value will be determined by the variables on the right-hand side.

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#### Model with *M* Independent Structural Equations (1)

 $Y_{1i} = \beta_{11} X_{1i} + \beta_{12} X_{2i} + \dots + \beta_{1K} X_{Ki} + \varepsilon_{1i} - -(1)$ 

 $Y_{Mi} = \beta_{M1} X_{1i} + \beta_{M2} X_{2i} + \dots + \beta_{MK} X_{Ki} + \varepsilon_{Mi} - -(M)$ 

#### Y - endogenously determined by X and $\boldsymbol{\epsilon}$

- X exogenously given
- $\boldsymbol{\epsilon}$  randomly determined with zero mean.

#### Model with *M* Independent Structural Equations (2)

 $\boldsymbol{\varepsilon}$  - uncorrelated between equations

 $E(\varepsilon) = 0$   $cov(\varepsilon_{mi}, \varepsilon_{nj}) = 0$ for all m, n = 1, ..., Mand all i, j==> equation-by-equation estimation (unrelated regression) is valid and BLUE

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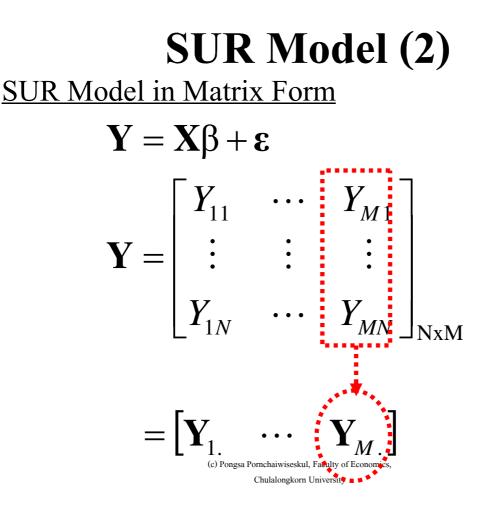
#### SUR Model (1)

Seeming Unrelated Regression Model

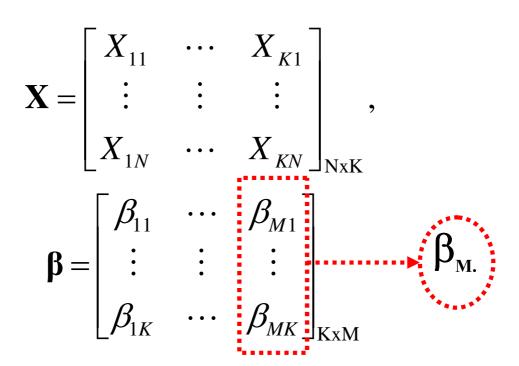
 $Y_{1i} = \beta_{11} X_{1i} + \beta_{12} X_{2i} + \dots + \beta_{1K} X_{Ki} + \varepsilon_{1i} - -(1)$ :

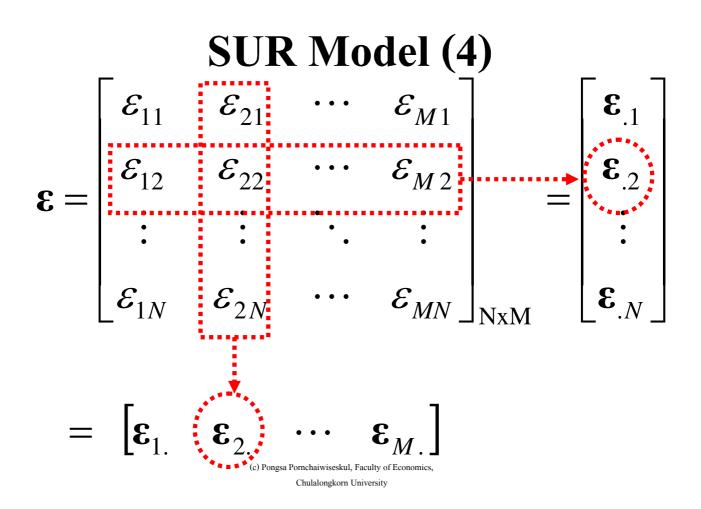
 $Y_{Mi} = \beta_{M1} X_{1i} + \beta_{M2} X_{2i} + \dots + \beta_{MK} X_{Ki} + \varepsilon_{Mi} - (M)$ 

$$\operatorname{cov}(\varepsilon_{mi}, \varepsilon_{ni}) \neq 0$$
 for some  $m \neq n$   
but  $\operatorname{cov}(\varepsilon_{mi}, \varepsilon_{nj}) = 0$  for all  $i \neq j$ 

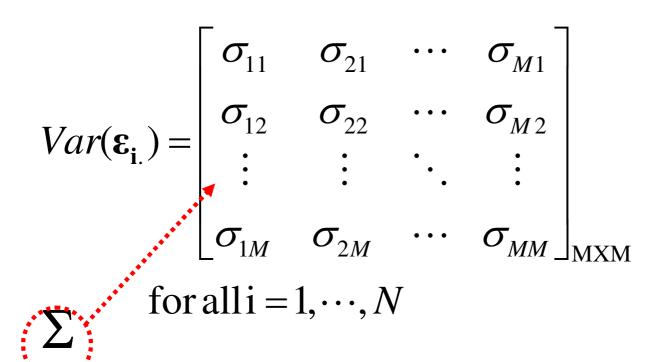


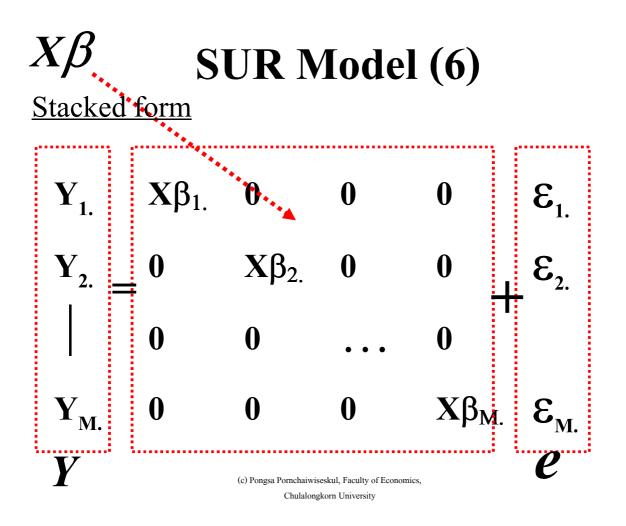
#### SUR Model (3)





#### SUR Model (5)





$$Var(\mathbf{e}) = \begin{bmatrix} \sigma_{11} & 0 & 0 & \sigma_{21} & 0 & 0 & \sigma_{M1} & \sigma_{M1} & 0 & 0 & \sigma_{M1} & \sigma_{M2} & \sigma_{M1} & \sigma_{M2} & \sigma_{M1} & \sigma_{M2} & \sigma_{M1} & \sigma_{M2} & \sigma_$$

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#### SUR Model (8)

Estimation Method

Step 0 Set  $\Sigma = I (MxM) \Rightarrow \Omega = I (MNxMN)$ Step 1 Run GLS on the "giant" stacked problem

$$\hat{\boldsymbol{\beta}} = [\boldsymbol{X}^{\mathrm{T}}(\hat{\boldsymbol{\Sigma}}^{-1} \otimes \boldsymbol{I}_{\mathrm{N}})\boldsymbol{X}]^{-1}\boldsymbol{X}^{\mathrm{T}}(\hat{\boldsymbol{\Sigma}}^{-1} \otimes \boldsymbol{I}_{\mathrm{N}})\boldsymbol{Y} \quad (\text{MKx1vector})$$

Step 2 Estimate  $\Sigma$  matrix

$$\hat{\sigma}_{mn} = \frac{\left[\mathbf{Y}_{m.} - \mathbf{X}\hat{\beta}_{m.}\right]^{\mathrm{T}}\left[\mathbf{Y}_{n.} - \mathbf{X}\hat{\beta}_{n.}\right]}{N}, \ m, n = 1, \dots, M$$

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#### SUR Model (9)

Estimation Method (cont'd) Step 3 Recalculate  $\Omega$   $\Omega^{T}\Omega = [\hat{\Sigma} \otimes \mathbf{I}_{N}]^{-1}$   $= \hat{\Sigma}^{-1} \otimes \mathbf{I}_{N}$   $\Omega = \hat{\Sigma}^{-\frac{1}{2}} \otimes \mathbf{I}_{N}$ Go back to step 1 until convergence is achieved. Note that SUR is also applied to Panel Data Analysis.

#### SUR with Cross-equation Restriction

#### <u>Linear Restriction</u> $R\beta = r$

In Step 1 above, starting from the stacked form  $Y=X\beta+e$ , apply weight matrix  $\Omega$ 

#### $= > \Omega Y = \Omega X \beta + \Omega e$

Apply RLS to the weighted stacked form. See "Inference" module for RLS.

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**SEM with Cross-equation Correlation** (1)  $Y_{1i} \neq \gamma_{12}Y_{2i} + \dots + \gamma_{1M}Y_{Mi}$  $+ \beta_{11}X_{1i} + \beta_{12}X_{2i} + \cdots + \beta_{1K}X_{Ki} + \varepsilon_{1i}$ •  $\gamma \neq 0$  $Y_{Mi} = \gamma_{M1}Y_{1i} + \dots + \gamma_{M,M-1}Y_{M-1,i}$  $+ \beta_{M1} X_{1i} + \beta_{M2} X_{2i} + \dots + \beta_{MK} X_{Ki} + \varepsilon_{Mi}$ Y - endogenously determined by X and  $\boldsymbol{\varepsilon}$ X - exogenously given (c) Pongsa Pornchaiwiseskul, Faculty of Economics,

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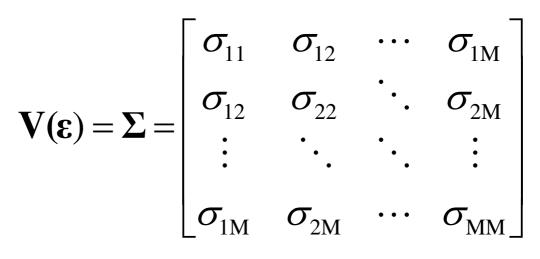
#### SEM with Cross-equation Correlation (2)

- $\boldsymbol{\varepsilon}$  randomly determined with zero mean.
- $\boldsymbol{\varepsilon}$  uncorrelated between equations

 $E(\varepsilon) = 0$   $cov(\varepsilon_{mi}, \varepsilon_{nj}) = 0$ for all m, n = 1, ..., Mand all i, j

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#### SEM with Cross-equation Correlation (3)



=> equation-by-equation estimation is <u>invalid</u>. Why?

#### SEM with Cross-equation Correlation (4)

Assume that X is non-random or

 $\operatorname{cov}(X_{ki}, \varepsilon_{mi}) = 0$  for all m = 1, ..., Mk = 1, ..., K and all i = 1, ..., N

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#### SEM with Cross-equation Correlation (5)

 $cov(Y_{mi}, \varepsilon_{ni}) = cov(\dots + \gamma_{mn}Y_{ni} + \dots + \varepsilon_{mi}, \varepsilon_{ni})$   $= \dots + \gamma_{mn} cov(Y_{ni}, \varepsilon_{ni}) + \dots + cov(\varepsilon_{mi}, \varepsilon_{ni})$   $= \dots + \gamma_{mn} cov(\dots + \varepsilon_{ni}, \varepsilon_{ni}) + \dots + cov(\varepsilon_{mi}, \varepsilon_{ni})$   $= \dots + \gamma_{mn} var(\varepsilon_{ni}) + \dots + cov(\varepsilon_{mi}, \varepsilon_{ni})$ That is,  $cov(Y_{mi}, \varepsilon_{ni}) \neq 0$ , in general, for all  $m \neq n$ . ==> violation of CLRM assumption

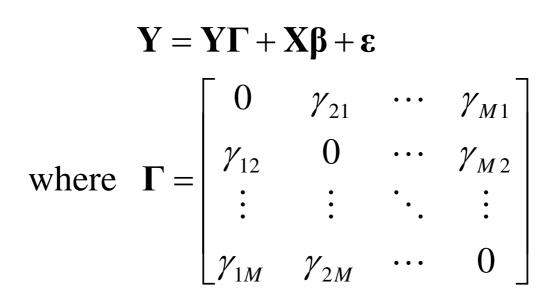
#### SEM with Cross-equation Correlation (6)

Valid Estimation Methods

- equation-by-equation 2SLS. Require a set of instrument variables(IV's) which include all the X's plus outside variables (if needed). Endogenous variables (Y's) cannot be IV.
- Simultaneous Approaches. Estimate all the equations simultaneously.

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#### **SEM in Matrix Form**



# **Identification Problems (1)**

No matter which approach is to be used (eq-by-eq or simultaneous), this question need to be answered first.

Can all the unknown parameters  $(\gamma, \beta)$ in each equation be <u>validly</u> estimated? What could be the problems?

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# **Identification Problems (2)**

The number of parameters to be estimated could exceed the number of available IV or exogenous variables.

Check Identification problems

- Exact Identification ==> proceed
- Over Identification ==> proceed
- Under Identification ==>need extra IV's

# **Identification Problems (3)**

Identification Problem need to be checked equation by equation. That is, one equation could be exactly identified while the other is under-identified.

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# **Identification Problems (4)**

- M = Max. number of endogenous variables in any equation (it is the number of eq's)
- K = Max. number of exogenous variables in any equation
- $m_m$  = number of  $\gamma$  parameters in equation *m* needed to be estimated (max. is M-1)
- $k_m$  = number of  $\beta$  parameters in equation *m* needed to be estimated (max. is K)

### **Identification Problems (5)**

#### Order Conditions

Equation *m* is

exactly identified if  $K = m_m + k_m$ over-identified if  $K > m_m + k_m$ under identified if  $K < m_m + k_m$ 

That is, check for # of  $(\gamma, \beta)$  parameters to be estimated in each equation against K.

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# **Identification Problems (6)**

 $\frac{\text{Rank Conditions}}{\text{Define } U_m = \text{sub-matrix of } \begin{bmatrix} \mathbf{I} - \mathbf{\Gamma} \\ -\mathbf{\beta} \end{bmatrix} \text{ which}$ includes only the rows corresponding to Y and X not in equation *m*.

Equation *m* is

identified if unidentified if  $Rank(U_m)=M-1$  $\operatorname{Rank}(U_m) < M-1$ 

## **Identification Problems (7)**

Equation-by-equation approach order condition is enough Simultaneous approach order condition is necessary rank condition is sufficient

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# **Identification Problems (8)**

#### ID Problem with intra-equation parameter restriction

 $r_m =$  number of non-redundant restriction on  $(\gamma, \beta)$  parameters in equation *m*, e.g.,

$$\gamma_{m2} + \beta_{m3} = 1$$
  

$$\beta_{m2} = \beta_{m4}$$
  
Check (m<sub>m</sub> + k<sub>m</sub> - r<sub>m</sub>) against K.  
Cross-eq restriction =>eqxeq method fails  
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#### **2SLS (1)**

The Same Two-Stage Least Square as in IV section. It is used eqn-by-eqn estn. Assumption

- all the X's are uncorrelated with all the ε's.
- 2) there could be extra IV's from outside model.

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# 2SLS (2)

For simplicity's sake, all IV could be assumed to be non-correlated with the error vector (all error terms in all the equation). More complex if some variables are correlated with some error terms but not with the others, e.g.,

 $Cov(X_{2i}, \varepsilon_{3i}) = 0$  but  $Cov(X_{2i}, \varepsilon_{4i}) \neq 0$ 

#### 2SLS (3)

Define

 $Z_{m} = matrix of IV's for equation m$ That is,  $cov(Z_{m}, \mathcal{E}_{m}) = 0$ <u>Stage 1</u> For each equation m, regress  $Y_{m}$ on  $Z_{m}$  using OLS. Get fitted value of  $Y_{m}$ .

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# 2SLS (4) $A_{m} = Z_{m} [Z_{m}^{T} Z_{m}]^{-1} Z_{m}^{T}, A_{m} = A_{m}^{T} = A_{m}^{2}$ $\hat{Y}_{m.} = A_{m} Y_{m.}$ $\underline{Stage 2} \text{ Substitute } Y_{m.} \text{ with } \hat{Y}_{m.} \text{ Then,}$ $estimate \gamma_{m.}, \beta_{m.} \text{ with OLS}$ Estimate

$$\hat{\boldsymbol{\Sigma}} = \frac{\begin{bmatrix} \mathbf{I} - \hat{\boldsymbol{\Gamma}}^{\mathrm{T}} & -\hat{\boldsymbol{\beta}}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{Y}^{\mathrm{T}} \\ \mathbf{X}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{Y} & \mathbf{X} \end{bmatrix} \begin{bmatrix} \mathbf{I} - \hat{\boldsymbol{\Gamma}} \\ -\hat{\boldsymbol{\beta}} \end{bmatrix}}{\underbrace{\mathbf{N}}_{\text{(c) Pongsa Pornchaiwiseskul, Faculty of Economics,}}}$$

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# Indirect Least Square (1)

Or **ILS** for short.

 $Y[I - \Gamma] = X\beta + \varepsilon$   $Y = X\beta[I - \Gamma]^{-1} + \varepsilon[I - \Gamma]^{-1}$   $= X\Pi + \xi \quad \leftarrow \text{Reduced Form}$ where  $\Pi = \beta[I - \Gamma]^{-1}$  KxM matrix  $\xi = \varepsilon[I - \Gamma]^{-1}$ 

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# **Indirect Least Square (2)**

Note that

- 1)  $Var(\xi)$  is not diagonal but  $Var(\varepsilon)$  is.
- 2) **Var**( $\xi$ ) =  $[I \Gamma^T]^{-1}\Sigma [I \Gamma]^{-1}$
- 3) if  $\Sigma$  and  $\Gamma$  are known, use SUR with

$$\mathbf{\Omega} = \left[ \mathbf{V}(\boldsymbol{\xi}) \otimes \mathbf{I}_N \right]^{-\frac{1}{2}}$$

#### **Indirect Least Square (3)**

<u>Step 1</u> Run equation-by-equation OLS on the reduced form

 $\Rightarrow \hat{\Pi}$  and  $Var(\hat{\Pi})$ 

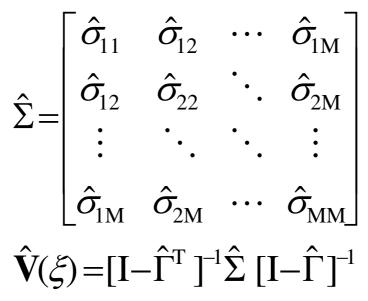
Unbiased but not the best for  $\Pi$ .

<u>Step 2</u> Estimate  $\beta$  and  $\Gamma$  matrices from  $\Pi$  estimates if they are identified.

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#### **Indirect Least Square (4)**

<u>Step 3</u> Calculate



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#### **Indirect Least Square (5)**

Step 4 Estimate the reduced form using SUR with  $\Omega$  weighting matrix.

$$\hat{\boldsymbol{\Omega}} = \left[ \hat{\mathbf{V}}(\boldsymbol{\xi}) \otimes \mathbf{I}_N \right]^{\frac{1}{2}}$$
$$= \hat{\mathbf{V}}(\boldsymbol{\xi})^{-\frac{1}{2}} \otimes \mathbf{I}_N$$

For Feasible ILS (FILS), go back to Step 2 until convergence

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## **Indirect Least Square (6)**

Note that

1) if they are identified ILS estimators are just valid (consistent) but not the best

2) Calculation of Var(  $\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\Gamma}}$ ) from

 $Var(\hat{\Pi})$  is rather complicated as their relationship is non-linear.

#### **Indirect Least Square (7)**

- 3) FILS coincides with 2SLS with only X's as IV.
- 4) ILS may be non-unique (overidentified). No over-identification problem for 2SLS.

=> ILS is much less popular than 2SLS.

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# **ID Problem Example (1)** Gujarati, p751

$$\begin{split} Y_{1t} &= \gamma_{12}Y_{2t} + \gamma_{13}Y_{3t} + \beta_{11} + \beta_{12}X_{2t} + u_{1t} \\ Y_{2t} &= \gamma_{23}Y_{3t} + \beta_{21} + \beta_{22}X_{2t} + \beta_{23}X_{3t} + u_{2t} \\ Y_{3t} &= \gamma_{31}Y_{1t} + \beta_{31} + \beta_{32}X_{2t} + \beta_{33}X_{3t} + u_{3t} \\ Y_{4t} &= \gamma_{41}Y_{1t} + \gamma_{42}Y_{2t} + \beta_{41} + \beta_{44}X_{4t} + u_{4t} \end{split}$$

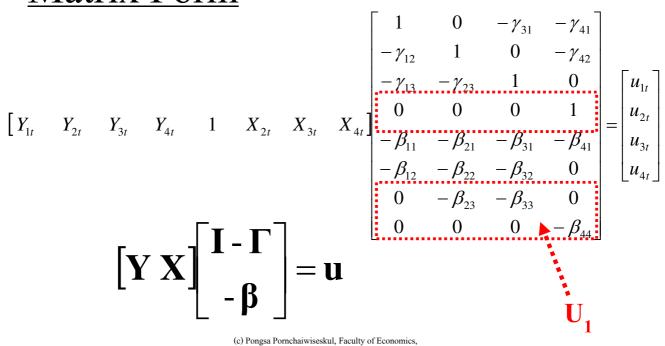
## **ID Problem Example (2)**

Note that

- there are 16 parameters in  $(\Gamma, \beta)$  to be estimated.
- There are also 16  $\pi$ 's in  $\Pi$  matrix
- Can  $(\Gamma, \beta)$  be identified from  $\Pi$  matrix? It seems OK. In fact, it is not.

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# **ID Problem Example (3)** <u>Matrix Form</u>



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#### **ID Problem Example (4)** Equation#1

$$U_{1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -\beta_{23} & -\beta_{33} & 0 \\ 0 & 0 & 0 & -\beta_{44} \end{bmatrix}$$
$$m_{1} + k_{1} = 4 = K \implies exact ?$$
$$Rank(U_{1}) = 2 \implies unidentified$$

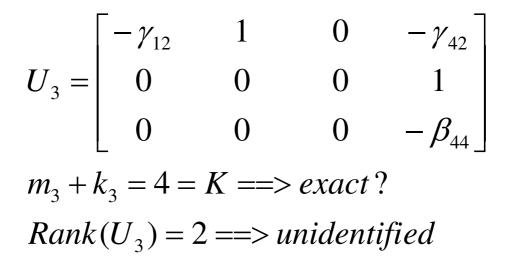
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#### **ID Problem Example (5)** Equation#2

$$U_{2} = \begin{bmatrix} 1 & 0 & -\gamma_{31} & -\gamma_{41} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\beta_{44} \end{bmatrix}$$
$$m_{2} + k_{2} = 4 = K \implies exact?$$
$$Rank(U_{2}) = 2 \implies unidentified$$

## **ID Problem Example (6)**

Equation#3



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#### **ID Problem Example (7)** Equation#4

$$U_{4} = \begin{bmatrix} -\gamma_{13} & -\gamma_{23} & 1 & 0 \\ -\beta_{12} & -\beta_{22} & -\beta_{32} & 0 \\ 0 & -\beta_{23} & -\beta_{33} & 0 \end{bmatrix}$$
$$m_{4} + k_{4} = 4 = K \implies exact?$$
$$Rank(U_{4}) = 3 \implies identified$$

#### **ID Problem Example (8)** 2SLS

All four equations will be identified with X's as IV (no IV from outside)

#### ILS

Only eq#4 will be identified. The other three are not completely identified. At least one parameter in each of these three equations will be unidentified according to the rank of  $U_m$  sub-matrices.

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## ID Problem Example (9)

If equation#2 is subject to an withinequation restriction, e.g.,

$$\begin{split} \gamma_{23} + \beta_{23} &= 1 \\ m_2 + k_2 - l_2 &= 3 < K \\ &==> over - identified ? \\ \text{Still,} \quad Rank(U_2) &= 2 ==> unidentified \end{split}$$

### **ID Problem Example (10)**

If equation#4 is subject to an withinequation restriction, e.g.,

$$\begin{split} \gamma_{41} &= \beta_{42} \\ m_4 + k_4 - l_4 &= 3 < K \\ &==> over - identified ? \\ Rank(U_4) &= 3 ==> over - identified \end{split}$$

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## **3SLS (1)**

- 3SLS or Three-Stage Least Square is the <u>simultaneous</u> extension of 2SLS to take advantage of SUR technique to improve the estimation efficiency.
- Note that in 2SLS the error terms are uncorrelated across equations. That is why equation-by-equation 2SLS is appropriate.

#### **3SLS (2)**

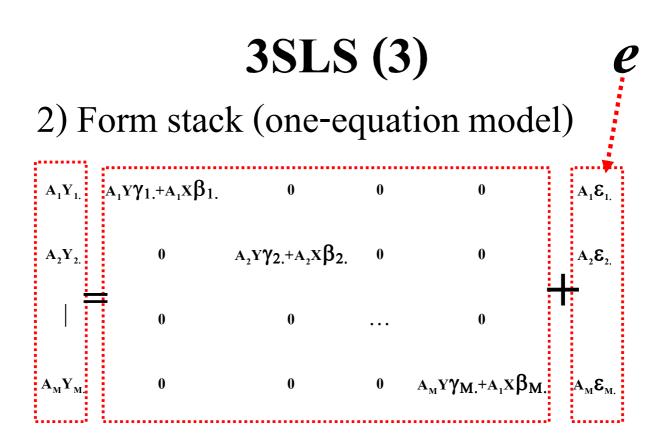
Start with the same  $\mathbf{Z}_{m}$  as in 2SLS.

Steps

0) Set  $\Sigma = I$ 

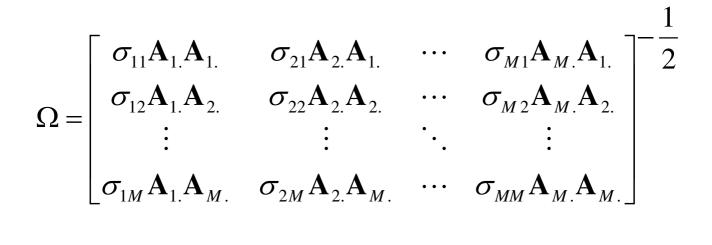
1) applying  $\mathbf{Z}_{m}$  as IV, the solution to equation *m* is equivalent to that of

 $\mathbf{A}_{m}\mathbf{Y}_{m} = \mathbf{A}_{m}\mathbf{Y}_{m}\mathbf{\Gamma}_{m} + \mathbf{A}_{m}\mathbf{X}\boldsymbol{\beta}_{m} + \mathbf{A}_{m}\boldsymbol{\varepsilon}_{m}$ Referred to IV section



#### **3SLS (4)**

#### 3) Estimate the weighting matrix $\Omega$



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#### **3SLS (5)**

3) Apply SUR to the stacked form using the estimated weight matrix.

==> valid estimates for  $\gamma$ , $\beta$  and  $\Sigma$ 

Note that there are restriction on  $\Gamma$ . Its diagonal elements are zero. Apply RLS.

Go back to 2) until convergence

# **GMM (1)**

#### Generalized Methods of Moments

\_extension over 3SLS by giving own weights and cross weights to each of IV to improve the estimation efficiency.

Same concept as GMM for one-equation model. Apply GMM to the stacked form which is now a single-equationed model.

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# **GMM (2)**

For each equation *m*, it is expected that

 $E(\mathbf{Z}_{m}(\mathbf{Y}_{m}[\mathbf{I}-\boldsymbol{\Gamma}_{m}]-\mathbf{X}\boldsymbol{\beta}_{m}))=\mathbf{0}$ for m=1,...,M Moment

#### Conditions

# **GMM (3)**

For simplicity, assume that all the equations share the same set of IV

$$Z_{m} = Z = [Z_{1}, Z_{2}, ..., Z_{S}]$$
  
for all m=1,...,M

That is,

$$E(\mathbf{Z}_{s}(\mathbf{Y}_{m}[I-\Gamma_{m.}]-X\beta_{m.}))=0 \text{ for all } s=1,...,S \text{ and } m=1,...,M$$

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# **GMM (4)**

Sample Analogy

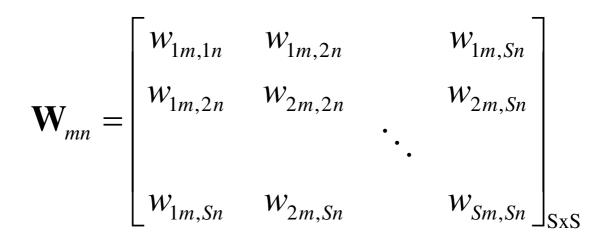
$$(\hat{\boldsymbol{\Gamma}}, \hat{\boldsymbol{\beta}})_{GMM} = \arg \min$$

$$\sum_{\substack{m=1\\n=1}}^{M} \sum_{\substack{s=1\\t=1}}^{S} \left( w_{sm,tn} \mathbf{Z}_{s}^{T} (\mathbf{Y}_{m.} - \mathbf{X} \boldsymbol{\beta}_{m.}) \mathbf{Z}_{t}^{T} (\mathbf{Y}_{n.} - \mathbf{X} \boldsymbol{\beta}_{n.}) \right)$$

where  $w_{sm,tn}$  = weight for the combination

# **GMM (5)**

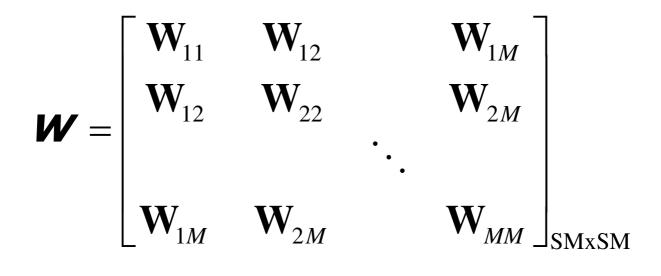
#### Put weights in the matrix form



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# **GMM (6)**

Put weights in the matrix form



# **GMM (7)**

Appropriate weight is

 $\hat{\boldsymbol{W}} = \left[ \hat{\boldsymbol{V}} [\boldsymbol{Z}^T (\boldsymbol{Y} [\boldsymbol{I} - \hat{\boldsymbol{\Gamma}}] - \boldsymbol{X} \hat{\boldsymbol{\beta}})]^{-1} \right]$ 

Estimation Step 1 W=I (SMxSM identity matrix) Step 2 Minimize

 $(\mathbf{Y}[\mathbf{I} - \boldsymbol{\Gamma}] - \mathbf{X}\boldsymbol{\beta})^T \mathbf{Z} \mathbf{W} \mathbf{Z}^T (\mathbf{Y}[\mathbf{I} - \boldsymbol{\Gamma}] - \mathbf{X}\boldsymbol{\beta})$ 

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#### **GMM (8)** Step 3 Estimate W for next iteration

$$\hat{\mathbf{W}} = \boldsymbol{\Sigma}^{-1}$$
$$= \frac{1}{N} (\mathbf{Y}[\mathbf{I} - \hat{\boldsymbol{\Gamma}}] - \mathbf{X}\hat{\boldsymbol{\beta}})^T (\mathbf{Y}[\mathbf{I} - \hat{\boldsymbol{\Gamma}}] - \mathbf{X}\hat{\boldsymbol{\beta}}) \otimes \mathbf{Z}^T \mathbf{Z}$$

Back to Step 2 until convergence occurs