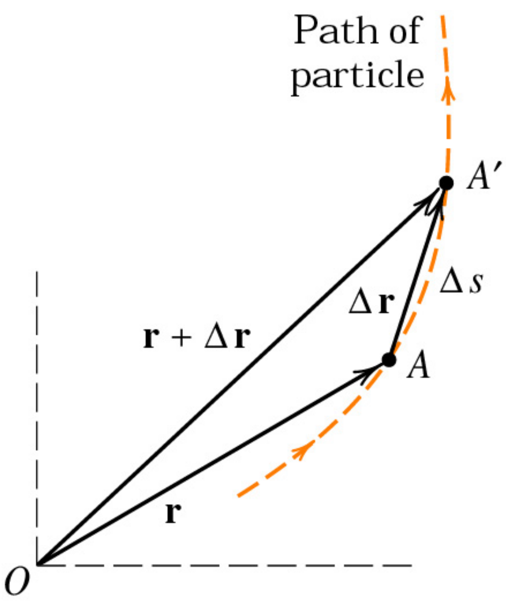
Plane curvilinear motion

Plane curvilinear motion

⇨ Motion of a particle along a curved path in a single plane



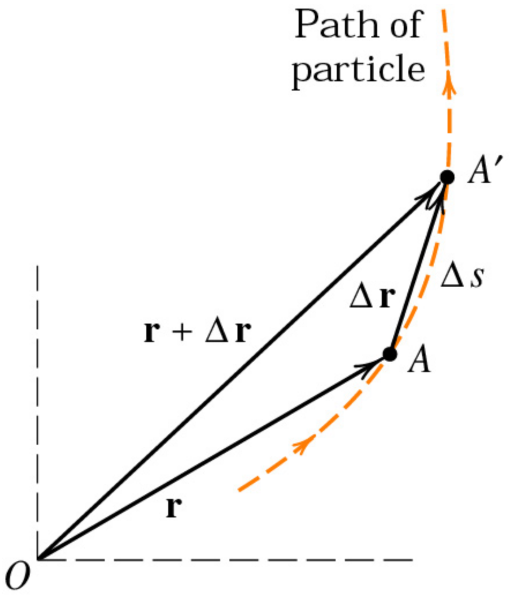
$\vec{r}$ = position vector (measured from some convenient fixed origin point O

$$A \rightarrow A' \qquad t \rightarrow t + \Delta t \qquad \vec{r} \rightarrow \vec{r} + \Delta \vec{r}$$

Displacement = vector change of position $\Delta \vec{r}$
(vector)

Distance = measured along the orange path Δs
(scalar)

Speed and velocity



Velocity $\vec{v} = \dfrac{d\vec{r}}{dt} = \dot{\vec{r}}$

Tangent to the path

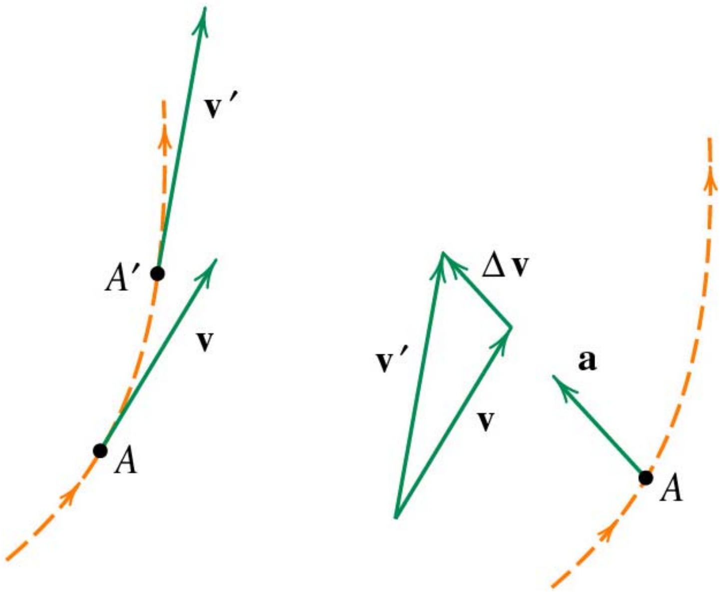
Speed $v = |\vec{v}| = \dfrac{ds}{dt} = \dot{s}$

*** Magnitude of derivative $\neq$ Magnitude of derivative

$$\left|\frac{d\vec{r}}{dt}\right| = |\vec{v}| = \dot{s}$$ ⇨ Speed

$$\frac{d|\vec{r}|}{dt} = \frac{dr}{dt} = \dot{r}$$ ⇨ Rate of change of the length of the vector $\vec{r}$

Acceleration



Average acceleration

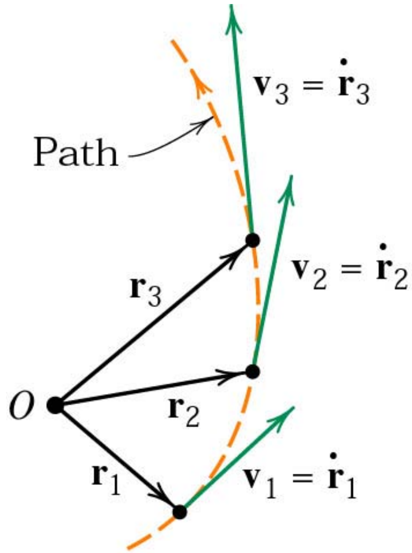
$$\vec{a}_{avg} = \frac{\Delta \vec{v}}{\Delta t}$$

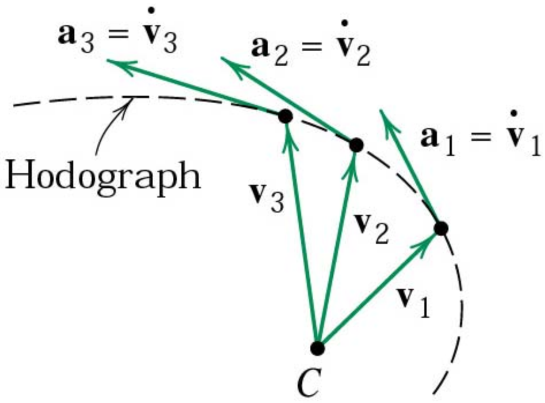
Acceleration

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} \Rightarrow \boxed{\vec{a} = \frac{d\vec{v}}{dt} = \dot{\vec{v}}}$$

- The velocity is always tangent to the path of the particle.
- The acceleration is <u>neither tangent</u> to the path <u>nor normal</u> to the path.

Visualization of motion





Path of a particle

Hodograph

- Path is constructed by motion vectors
- Velocity tangent to the path

- Path is constructed by velocity vectors
- Acceleration tangent to the path

Derivatives of vectors

Obey the same rules as they do for scalars

$$\frac{d\vec{P}}{dt} = \dot{P}_x\hat{i} + \dot{P}_y\hat{j} + \dot{P}_z\hat{k}$$

$$\frac{d\vec{P}u}{dt} = \vec{P}\dot{u} + \dot{\vec{P}}u$$

$$\frac{d(\vec{P}\cdot\vec{Q})}{dt} = \vec{P}\cdot\dot{\vec{Q}} + \dot{\vec{P}}\cdot\vec{Q}$$

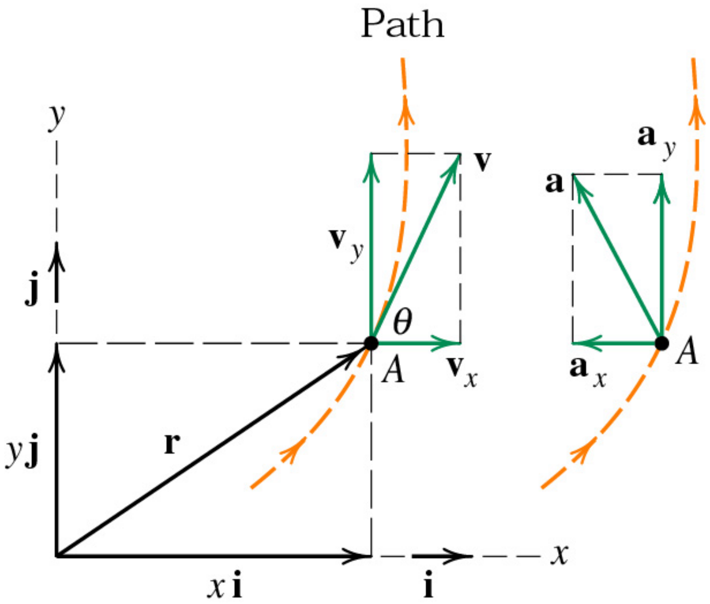
$$\frac{d(\vec{P}\times\vec{Q})}{dt} = \vec{P}\times\dot{\vec{Q}} + \dot{\vec{P}}\times\vec{Q}$$

The coordinate systems

- Rectangular, x-y
- Normal-tangent, n-t
- Polar, r-θ

Selection depended on the problem considered

Rectangular coordinate



Rectangular coordinates are usually good for problems where *x* and *y* variable can be calculated independently

$$\vec{r} = x\hat{i} + y\hat{j}$$

$$\vec{v} = \dot{\vec{r}} = \dot{x}\hat{i} + \dot{y}\hat{j}$$

$$\vec{a} = \dot{\vec{v}} = \ddot{\vec{r}} = \ddot{x}\hat{i} + \ddot{y}\hat{j}$$

Other relations

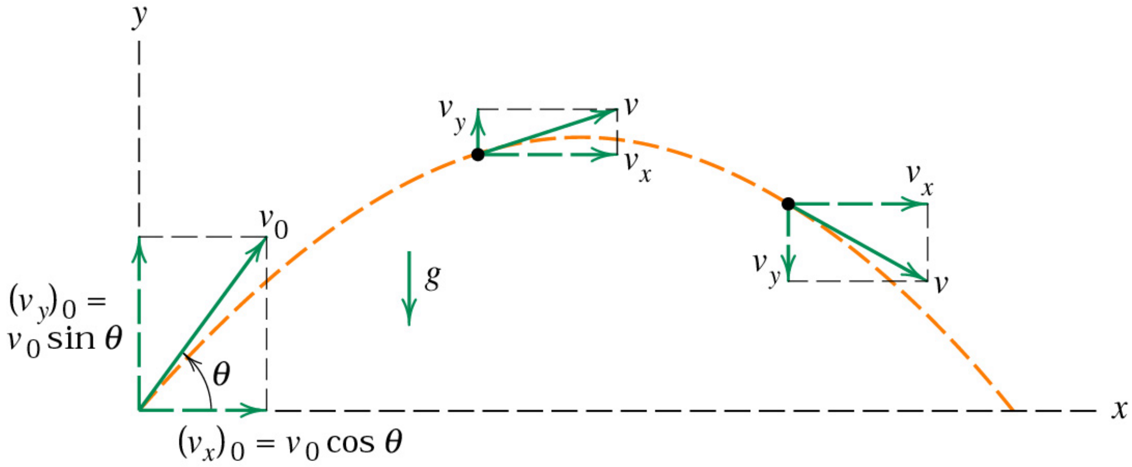
$$v^2 = v_x{}^2 + v_y{}^2$$

$$\tan\theta = \frac{v_y}{v_x}$$

$$a^2 = a_x{}^2 + a_y{}^2$$

Projectile motion

The most common case ⇨ $a_x = 0, \quad a_y = -g$



x and y direction can be calculated independently

<u>*x*-axis</u>

$$v_x = (v_x)_0$$

$$x = x_0 + (v_x)_0 t$$

<u>*y*-axis</u>

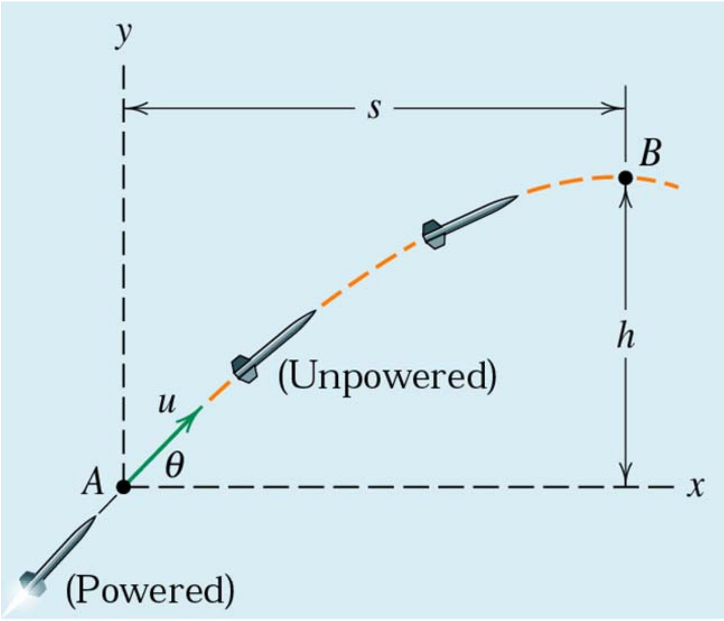
$$v_y = (v_y)_0 - gt$$

$$y = y_0 + (v_y)_0 t - \frac{1}{2} g t^2$$

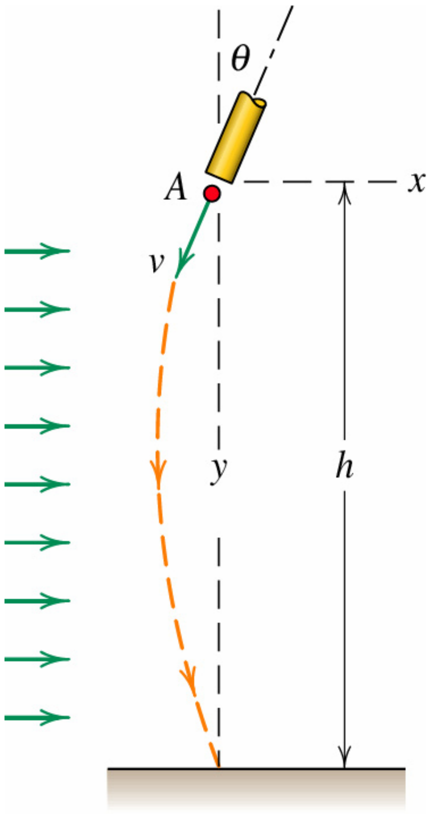
$$v_y^{\,2} = (v_y)_0^{\,2} - 2g(y - y_0)$$

Sample 1

A rocket has expended all its fuel when it reaches position *A*, where it has a velocity **u** at an angle θ with respect to the horizontal. It then begins unpowered flight and attains a maximum added height *h* at position *B* after traveling a horizontal distance *s* from *A*. Determine the expressions for *h* and *s*, the time *t* of flight from *A* to *B*, and the equation of the path. For the interval concerned, assume a flat earth with a constant gravitational acceleration *g* and neglect any atmospheric resistance.



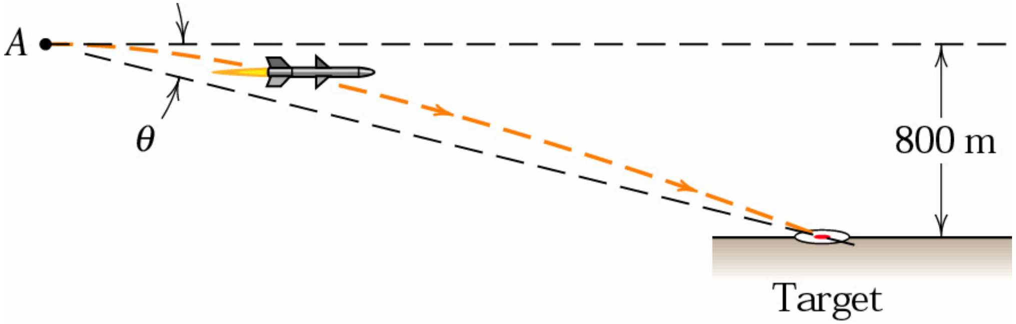
Sample 2 (2/73)



A particle is ejected from the tube at *A* with a velocity *v* at an angle *θ* with the vertical *y*-axis. A strong horizontal wind gives the particle a constant horizontal acceleration a in the *x*-direction. If the particle strikes the ground at a point directly under its released position, determine the height *h* of point *A*. The downward *y*-acceleration may be taken as the constant *g*.

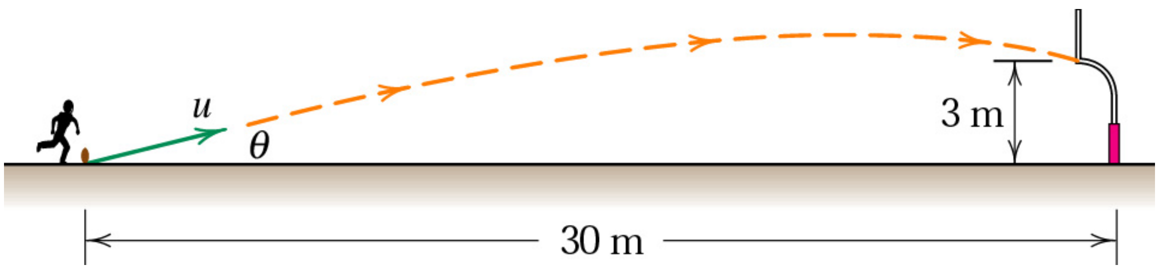
Sample 3 (2/77)

A rocket is released at point *A* from a jet aircraft flying horizontally at 1000 km/h at an altitude of 800 m. If the rocket thrust remains horizontal and gives the rocket a horizontal acceleration of 0.5*g*, determine the angle θ from the horizontal to the line of sight to the target.



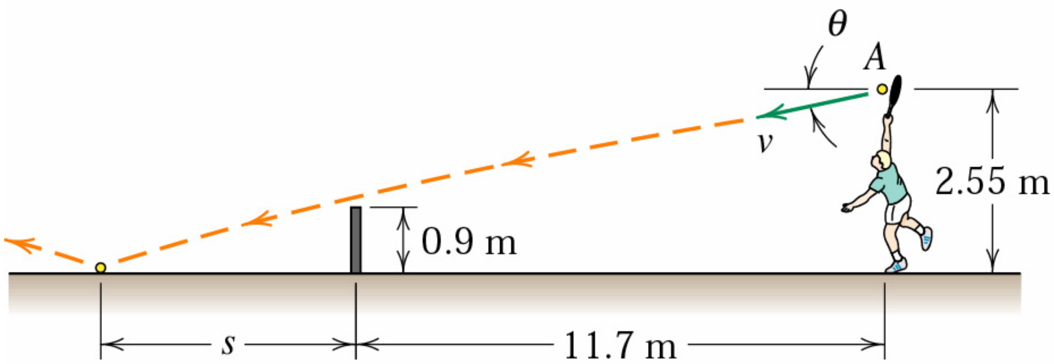
Sample 4 (2/83)

A football player attempts a 30-m field goal. If he is able to impart a velocity u of 30 m/s to the ball, compute the minimum angle θ for which the ball will clear the crossbar of the goal. (Hint: Let $m = \tan\theta$.)



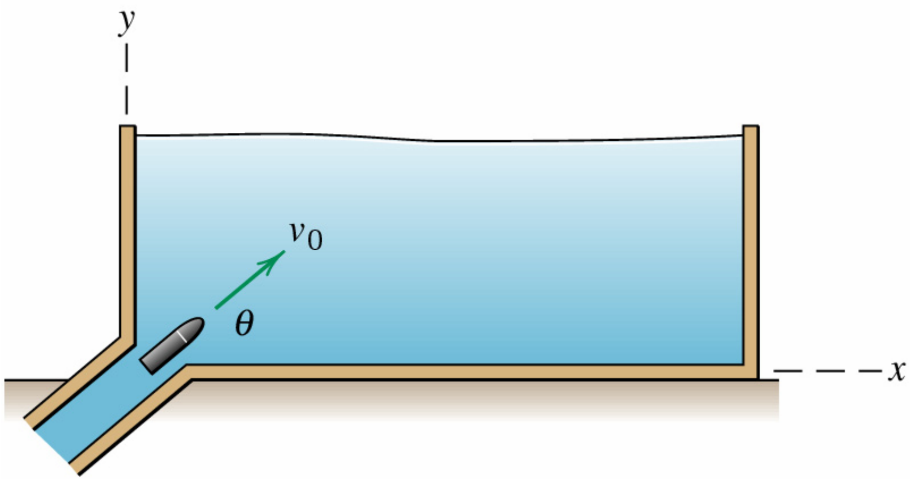
Sample 5 (2/85)

If the tennis player serves the ball horizontally ($\theta = 0$), calculate its velocity v if the center of the ball clears the 0.9-m net by 150 mm. Also find the distance s from the net to the point where the ball hits the court surface. Neglect air resistance and the effect of ball spin.



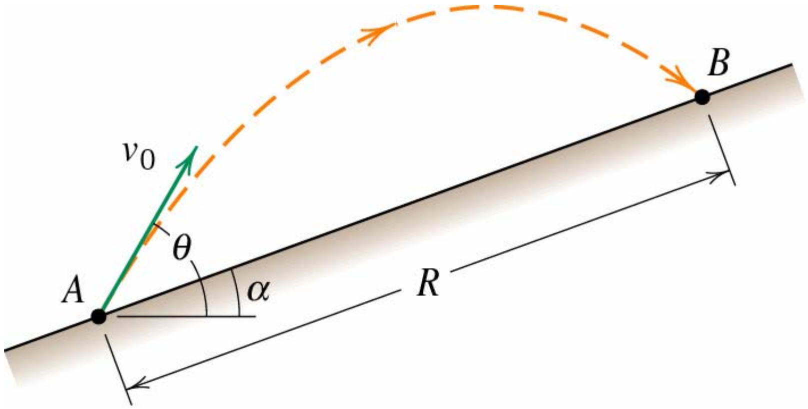
Sample 6 (2/94)

A projectile is ejected into an experimental fluid at time $t = 0$. The initial speed is v_0 and the angle to the horizontal is θ. The drag on the projectile results in an acceleration term $\mathbf{a}_D = -k\mathbf{v}$, where k is a constant and $\mathbf{v}$ is the velocity of the projectile. Determine the x- and y-components of both the velocity and displacement as functions of time. What is the terminal velocity? Include the effects of gravitational acceleration.

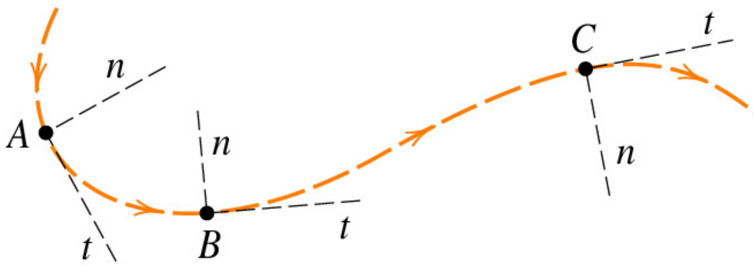


Sample 7 (2/95)

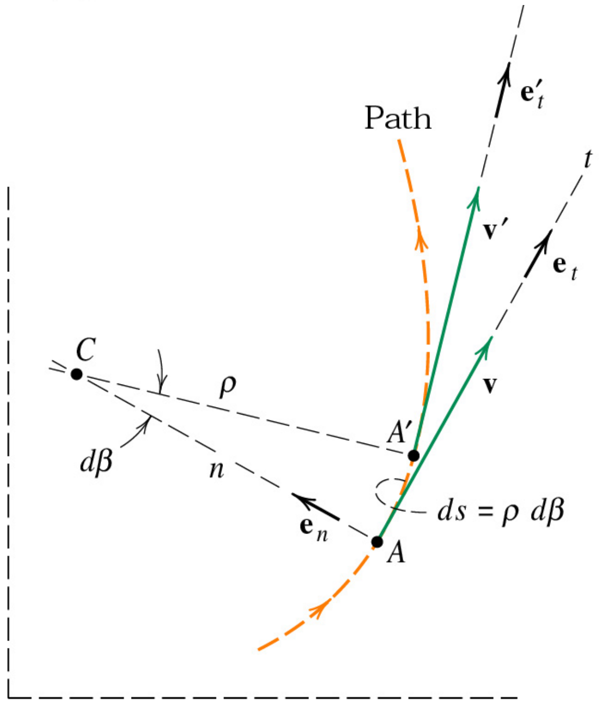
A projectile is launched with speed v_0 from point *A*. Determine the launch angle θ which results in the maximum range *R* up the incline of angle α (where $0 \leq \alpha \leq 90°$). Evaluate your results for a = 0, 30°, and 45 °.



Normal and tangential coordinates (*n*-*t*)



- Positive *n* toward the center of the curvature of the path
- Positive *t* in the direction of positive velocity
- The origin and axes move and rotate together with the particle

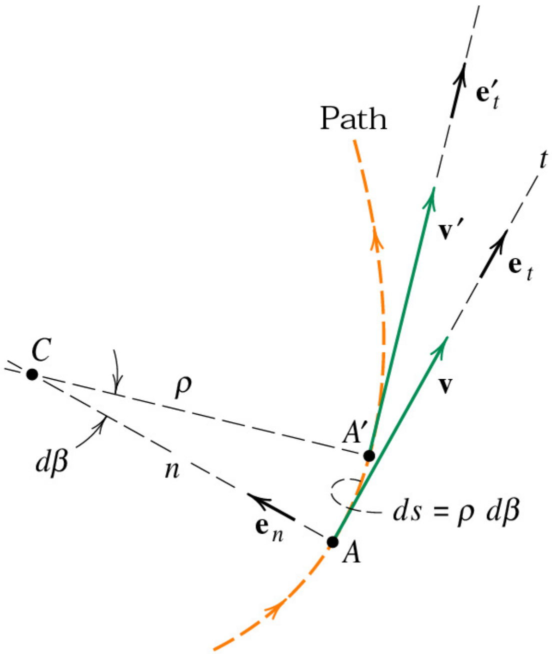


$\hat{e}_n$, $\hat{e}_t$: unit vector in the *n*- and *t*-direction

$$\vec{v} = v\hat{e}_t = \left(\frac{ds}{dt}\right)\hat{e}_t = \left(\frac{\rho d\beta}{dt}\right)\hat{e}_t = \rho\dot{\beta}\hat{e}_t$$

velocity is always tangent to the path ⇨ Only *t* component

Acceleration (1)

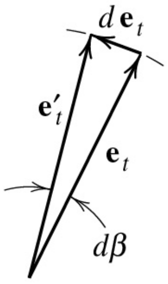


$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d(v\hat{e}_t)}{dt} = v\dot{\hat{e}}_t + \dot{v}\hat{e}_t$$

$$\dot{\hat{e}}_t = \frac{d\hat{e}_t}{dt} = \left(\frac{d\hat{e}_t}{d\beta}\right)\cdot\left(\frac{d\beta}{dt}\right)$$

$$\dot{\hat{e}}_t = \dot{\beta}\hat{e}_n$$

$$v = \rho\dot{\beta}$$

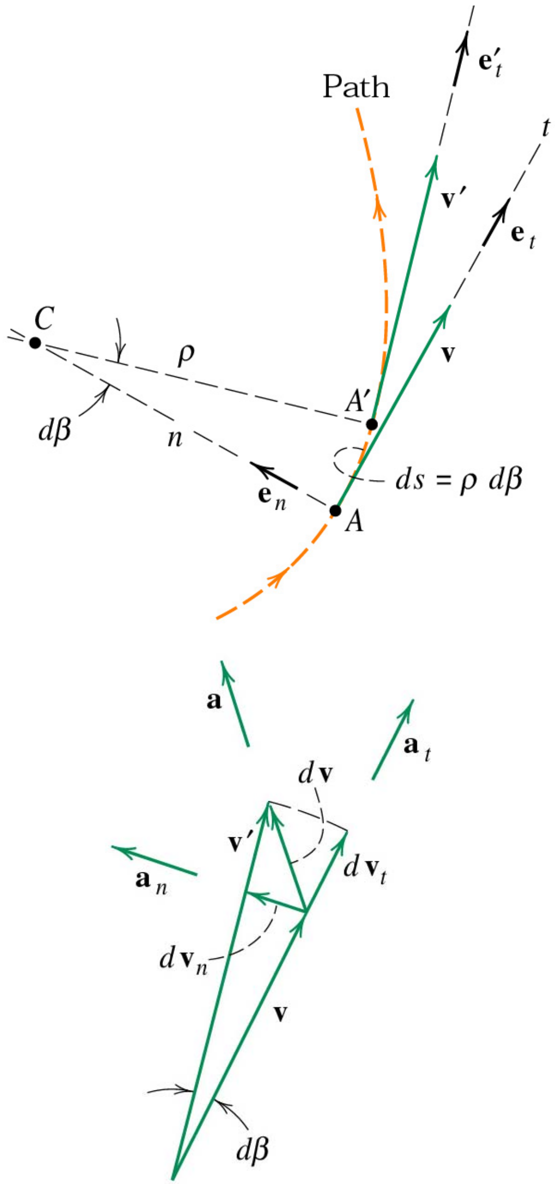


$$d\hat{e}_t = |\hat{e}_t| d\beta \cdot \hat{e}_n$$

$$\frac{d\hat{e}_t}{d\beta} = \hat{e}_n$$

$$\vec{a} = \frac{v^2}{\rho}\hat{e}_n + \dot{v}\hat{e}_t$$

Acceleration (2)



$$\vec{a} = \frac{v^2}{\rho}\hat{e}_n + \dot{v}\hat{e}_t$$

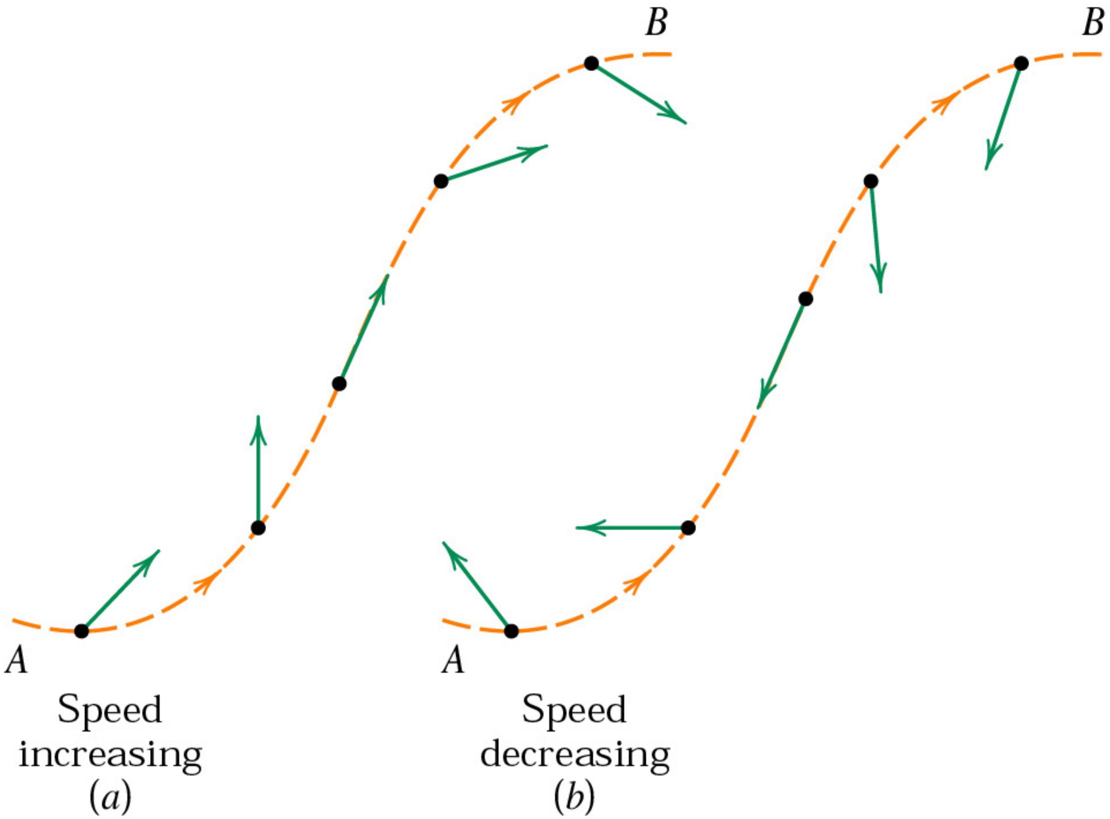
$$a_n = \frac{v^2}{\rho} = \rho\dot{\beta}^2 = v\dot{\beta}$$ Change in direction

$$a_t = \dot{v} = \ddot{s}$$ Change in amplitude

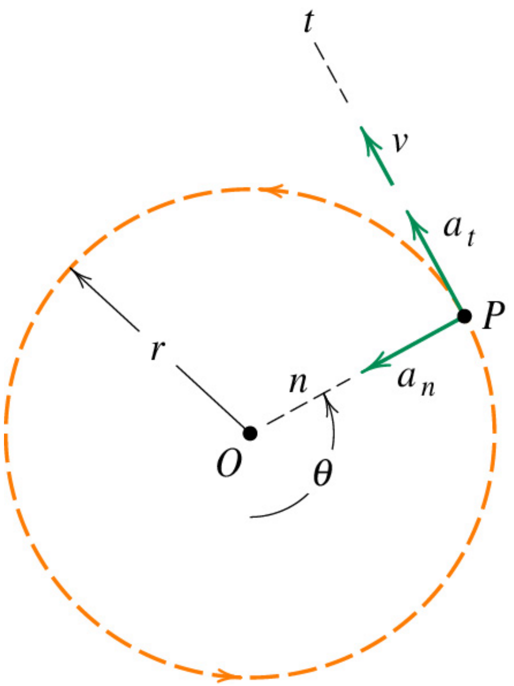
$$a = \sqrt{a_n^{\ 2} + a_t^{\ 2}}$$

Direction of acceleration

a_n is always directed toward the center of curvature



Circular motion



ρ ⇒ Constant r

β ⇒ θ

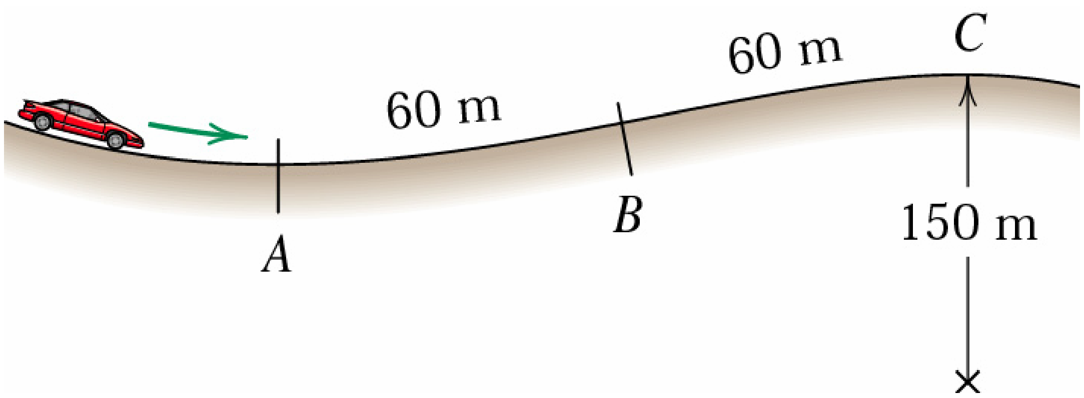
$$v = r\dot{\theta}$$

$$a_n = \frac{v^2}{r} = r\dot{\theta}^2 = v\dot{\theta}$$

$$a_t = \dot{v} = r\ddot{\theta}$$

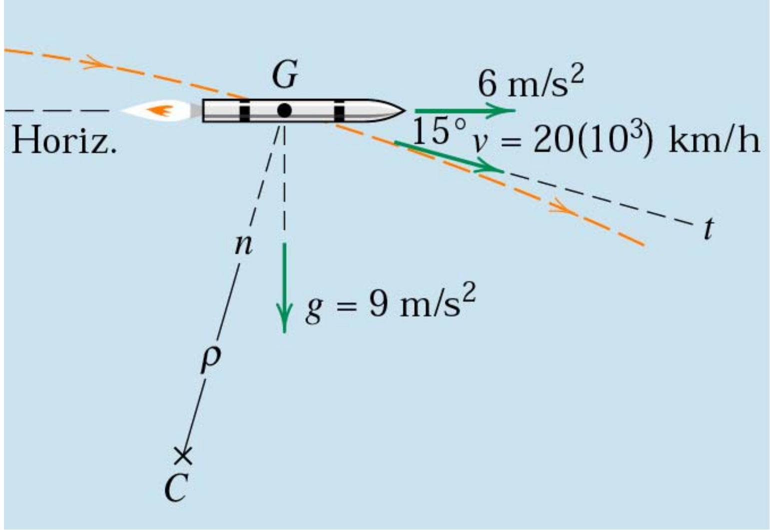
Sample problem 2/7

To anticipate the dip and hump in the road, the driver of a car applies her brakes to produce a uniform deceleration. Her speed is 100 km/h at the bottom *A* of the dip and 50 km/h at the top *C* of the hump, which is 120 m along the road from *A*. If the passengers experience a total acceleration of 3 m/s^2 at *A* and if the radius of curvature of the hump at *C* is 150 m, calculate (*a*) the radius of curvature ρ at *A*, (*b*) the acceleration at the inflection point *B*, and (*c*) the total acceleration at *C*.



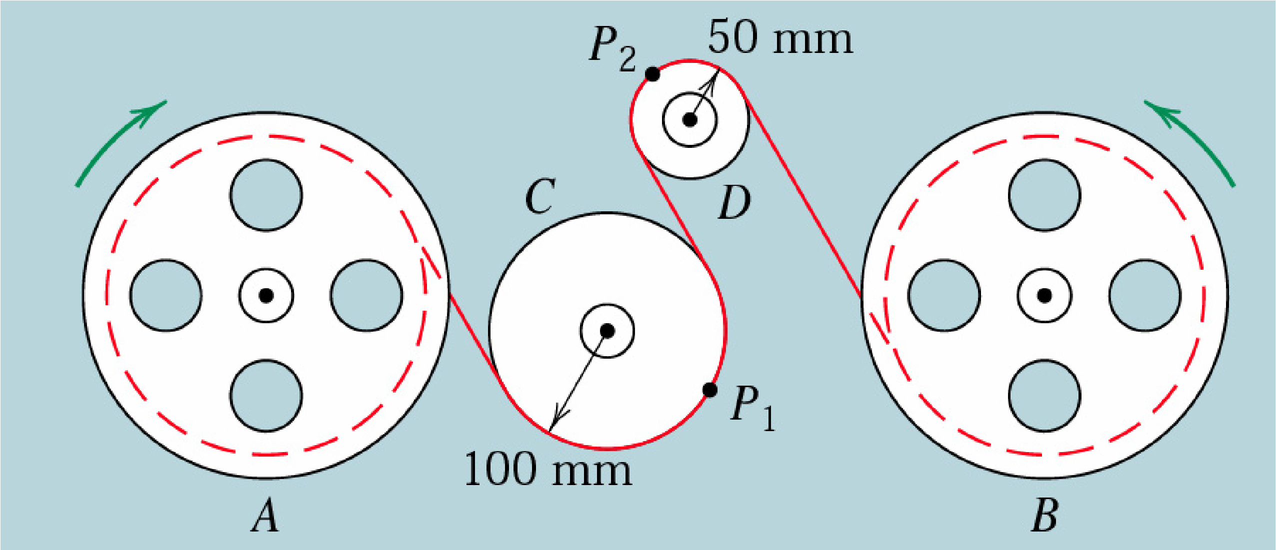
Sample problem 2/8

The thrust imparts a horizontal component of acceleration of 6 m/s^2, and the downward acceleration component is the acceleration due to gravity, which is g = 9 m/s^2. At the instant represented, the velocity of the mass center G of the rocket along the 15° direction of its trajectory is 20(10^3) km/h. Determine (a) the radius of curvature of the flight trajectory, (b) the rate at which the speed v is increasing, (c) the angular rate $\dot{\beta}$ of the radial line from G to the center of curvature C, and (d) the vector expression for the total acceleration a of the rocket.



Sample 8 (2/113)

From the figure, if the normal component of acceleration of P_1 is 40 m/s^2 and the tangential component of acceleration of P_2 is 30 m/s^2 at this instant, compute the corresponding speed v of the tape, the magnitude of the total acceleration of P_1, and the magnitude of the total acceleration of P_2.



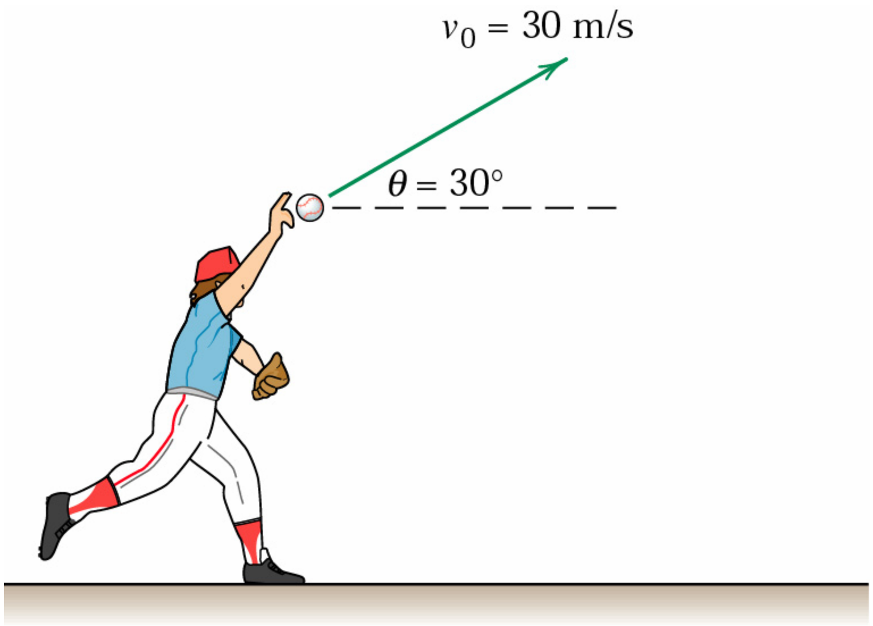
Sample 9 (2/119)

A particle moving in the *x*-*y* plane has a position vector given by $\vec{r} = \frac{3}{2}t^2\hat{i} + \frac{2}{3}t^3$, where $\vec{r}$ is in meters and *t* is in seconds.

Calculate the radius of curvature *r* of the path for the position of the particle when *t* = 2 s. Sketch the velocity *v* and the curvature of the path for this particular instant.

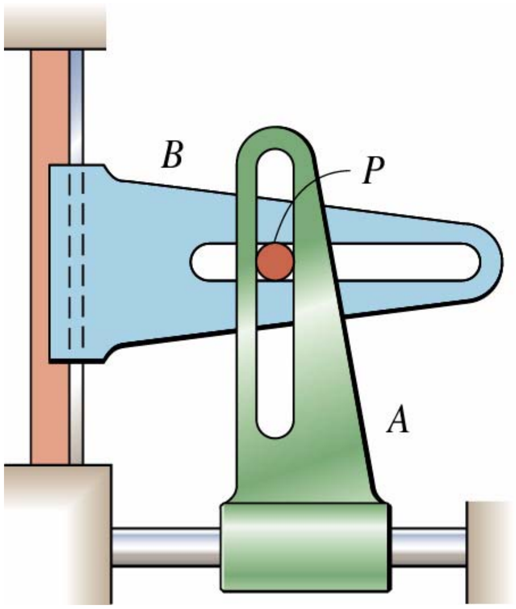
Sample 10 (2/121)

A baseball player releases a ball with the initial conditions shown in the figure. Determine the radius of curvature ρ of the path and the time rate of change $\dot{v}$ of the speed at times $t = 1$ s and $t = 2.5$ s, where $t = 0$ is the time of release from the player's hand.



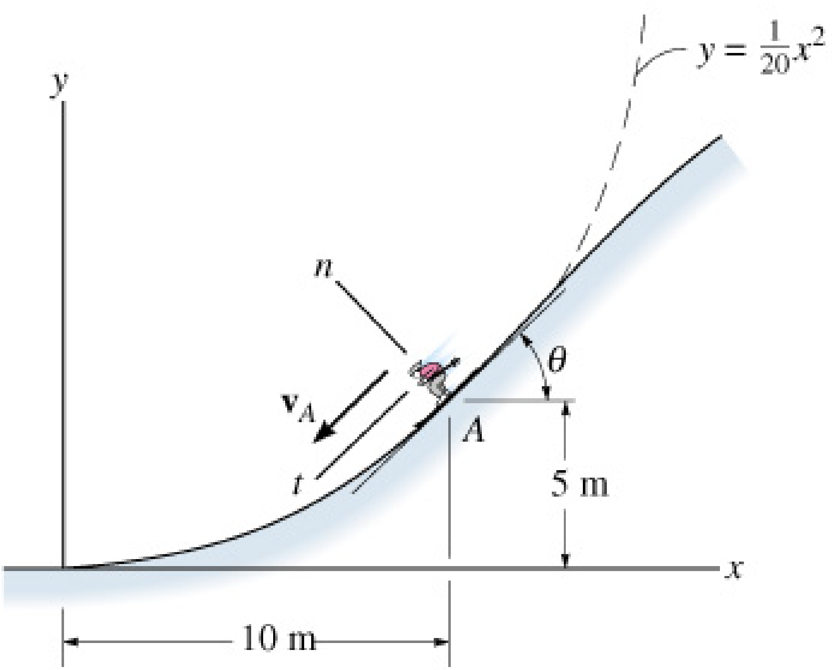
Sample 11 (2/128)

At the instant represented, *A* has a velocity to the right of 0.2 m/s which is decreasing at the rate of 0.75 m/s each second. At the same time, *B* is moving down with a velocity of 0.15 m/s which is decreasing at the rate of 0.5 m/s each second. For this instant determine the radius of curvature ρ of the path followed by *P*. Is it possible to determine also the time rate of change of ρ?



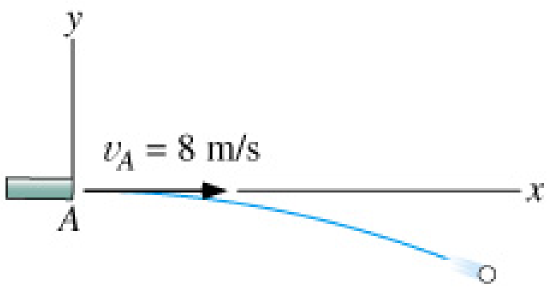
Sample 12

When the skier reaches point A along the parabolic path, he has a speed of 6 m/s which is increasing at 2 m/s2. Determine the direction of his velocity and the direction and magnitude of his acceleration at this instant. Neglect the size of the skier in the calculation.

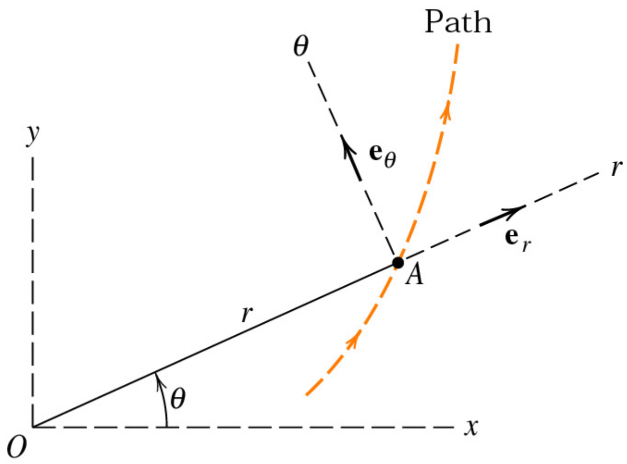


Sample 13

The ball is ejected horizontally from the tube with a speed of 8 m/s. Find the equation of the path, y = f(x), and then find the ball's velocity and the normal and tangential component of acceleration when t = 0.25 s.



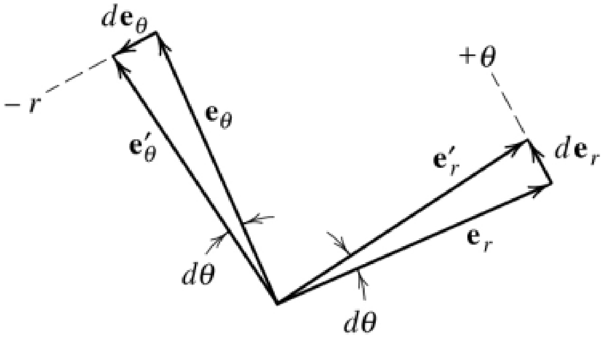
Polar coordinates (r-θ)



r and θ are measured from an arbitrary fixed reference (x-y axes)

$$\vec{r} = r\hat{e}_r$$

Time derivative of the unit vectors



$$d\hat{e}_r = \left|\hat{e}_r\right| \cdot d\theta \cdot \hat{e}_\theta \quad (\left|\hat{e}_r\right| = 1)$$

$$\dot{\hat{e}}_r = \dot{\theta} \cdot \hat{e}_\theta$$

$$d\hat{e}_\theta = \left|\hat{e}_\theta\right| \cdot d\theta \cdot (-\hat{e}_r) \quad (\left|\hat{e}_\theta\right| = 1)$$

$$\dot{\hat{e}}_\theta = -\dot{\theta} \cdot \hat{e}_r$$

Velocity and acceleration

Basic relations $\vec{r} = r\hat{e}_r\,, \qquad \dot{\hat{e}}_r = \dot{\theta}\cdot\hat{e}_\theta\,, \qquad \dot{\hat{e}}_\theta = -\dot{\theta}\cdot\hat{e}_r$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{e}_r + r\dot{\hat{e}}_r$$

$$\vec{v} = \dot{r}\hat{e}_r + (r\dot{\theta})\hat{e}_\theta$$

$v_r = \dot{r}$ (caused by the change of the length of $\vec{r}$)

$v_\theta = r\dot{\theta}$ (caused by the rotation of $\vec{r}$)

$$v = \sqrt{v_r{}^2 + v_\theta{}^2}$$

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$= (\ddot{r}\hat{e}_r + \dot{r}\dot{\hat{e}}_r) + (\dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\dot{\hat{e}}_\theta)$$

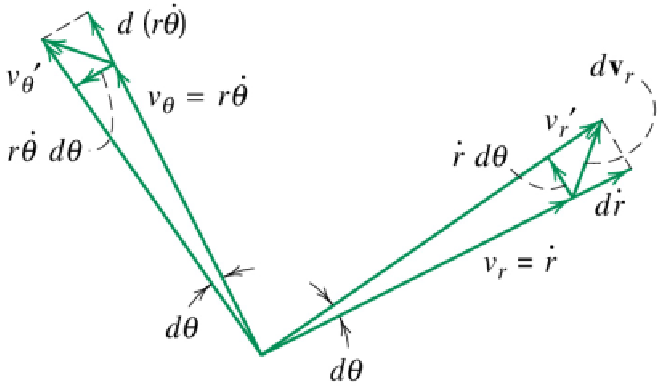
$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

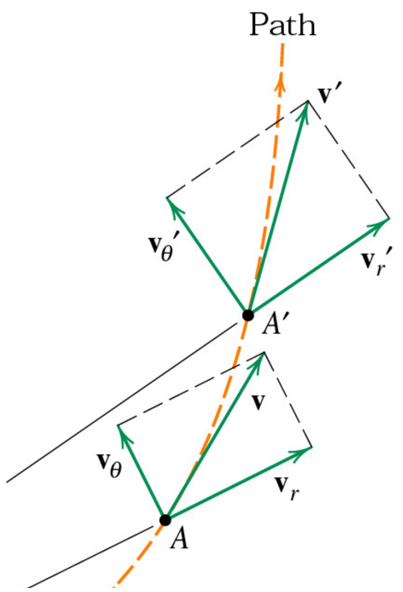
$$a_r = \ddot{r} - r\dot{\theta}^2$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

$$a = \sqrt{a_r{}^2 + a_\theta{}^2}$$

Geometric interpretation





$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

Magnitude change of $\vec{v}_r$ $= d\dot{r}$

⇒ $\ddot{r}$ (in r direction)

Direction change of $\vec{v}_r$ $= \dot{r}d\theta$

⇒ $\dot{r}\dot{\theta}$ (in θ direction)

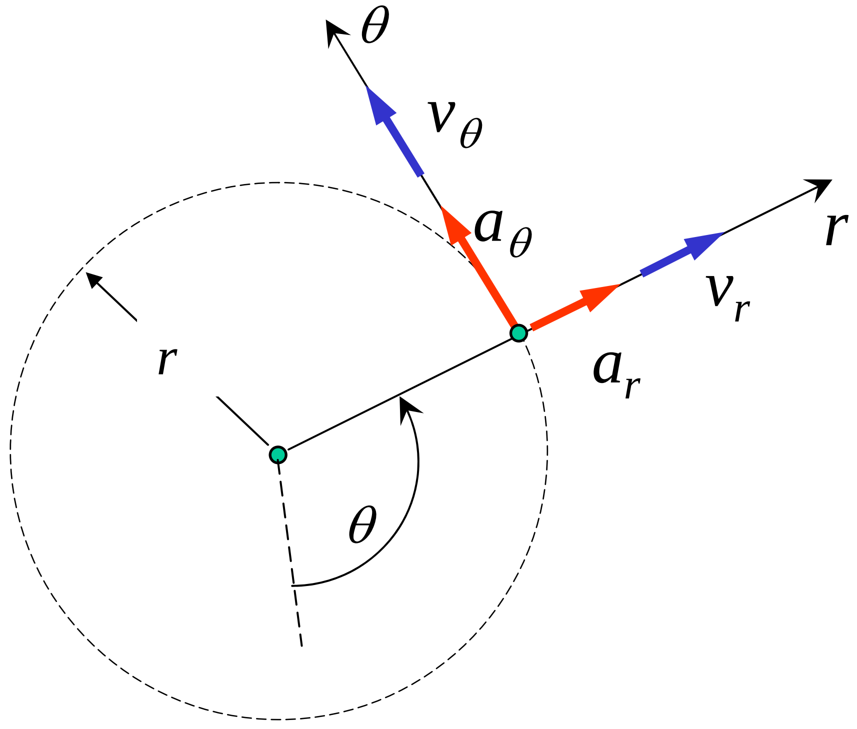
Magnitude change of $\vec{v}_\theta$ $= d(r\dot{\theta})$

⇒ $\dfrac{d(r\dot{\theta})}{dt} = \dot{r}\dot{\theta} + r\ddot{\theta}$ (in θ direction)

Direction change of $\vec{v}_\theta$ $= r\dot{\theta}d\theta$

⇒ $r\dot{\theta}^2$ (in $-r$ direction)

Circular motion (r-θ)



$$\vec{v} = \dot{r}\hat{e}_r + (r\dot{\theta})\hat{e}_\theta$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

$$v_r = \dot{r} = 0$$

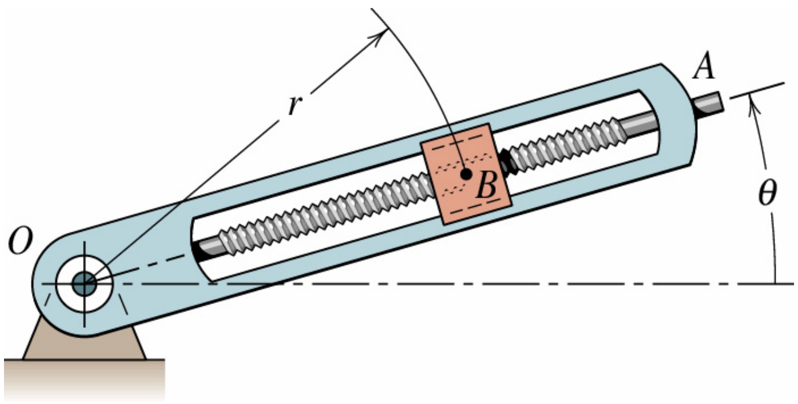
$$v_\theta = r\dot{\theta}$$

$$a_r = \ddot{r} - r\dot{\theta}^2 = -r\dot{\theta}^2 = -a_n$$

$$a_\theta = r\ddot{\theta}$$

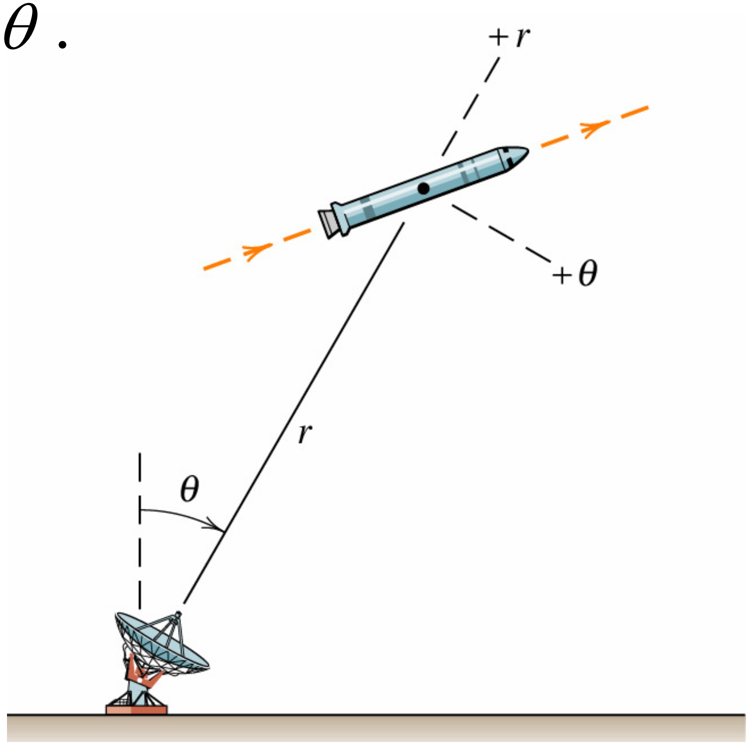
Sample problem 2/9

Rotation of the radially slotted arm is governed by $\theta = 0.2t + 0.02t^3$, where θ is in radians and t is in seconds. Simultaneously, the power screw in the arm engages the slider B and controls its distance from O according to $r = 0.2 + 0.04t^2$, where r is in meters and t is in seconds. Calculate the magnitudes of the velocity and acceleration of the slider for the instant when $t = 3$ s.



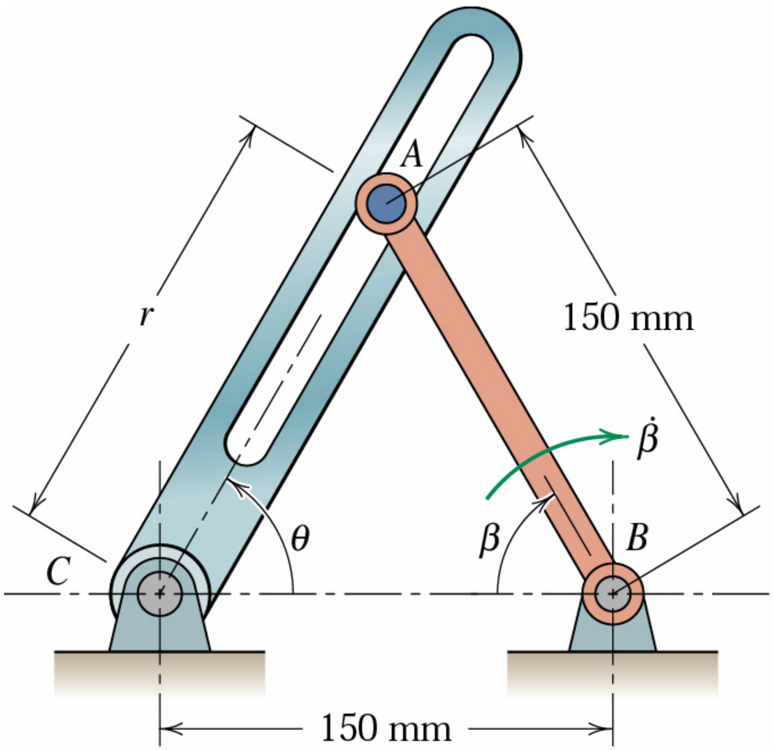
Sample problem 2/10

A tracking radar lies in the vertical plane of the path of a rocket which is coasting in unpowered flight above the atmosphere. For the instant when $\theta = 30°$, the tracking data give $r = 8(10^4)$ m, $\dot{r} =$ 1200 m/s, and $\dot{\theta} = 0.80$ deg/s. The acceleration of the rocket is due only to gravitational attraction and for its particular altitude is 9.20 m/s^2 vertically down. For these condition determine the velocity v of the rocket and the values of $\ddot{r}$ and $\ddot{\theta}$.



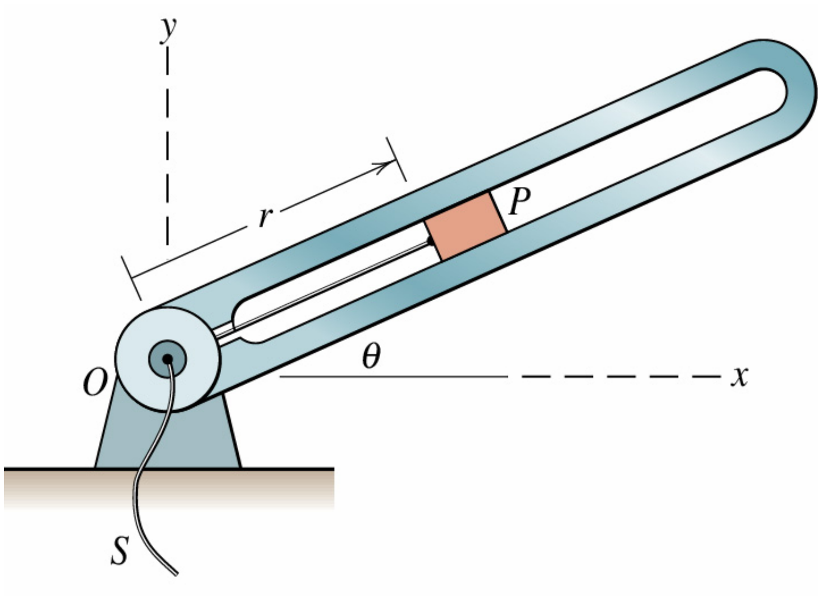
Sample 12 (2/141)

Link *AB* rotates through a limited range of angle β, and its end *A* causes the slotted link *AC* to rotate also. For the instant represented where $\beta = 60$ and $\dot{\beta} = 0.6$ rad/s constant, determine the corresponding value of $\dot{r}$, $\ddot{r}$, $\dot{\theta}$, and $\ddot{\theta}$. Make use of Eqs. 2/13 and 2/14.



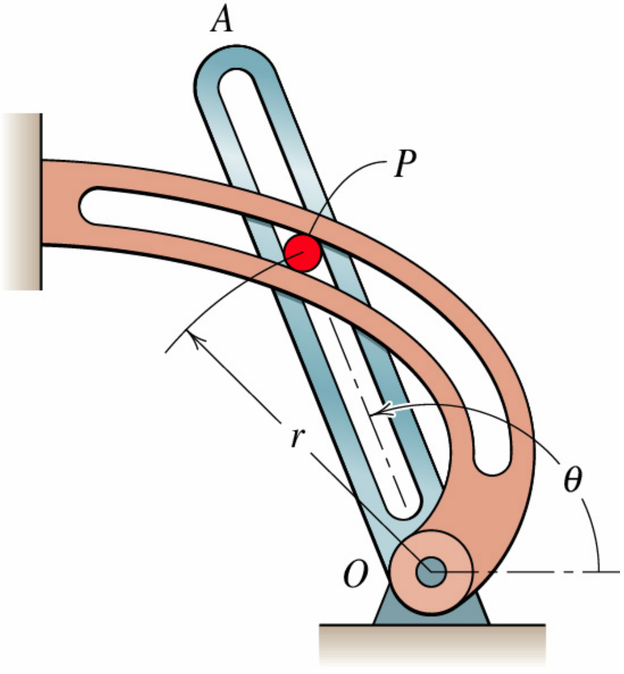
Sample 13 (2/145)

The slider *P* can be moved inward by means of the string *S*, while the slotted arm rotates about point *O*. The angular position of the arm is given by $\theta = 0.8t - t^2/20$, where θ is in radians and *t* is in seconds. The slider is at *r* = 1.6 m when *t* = 0 and thereafter is drawn inward at the constant rate of 0.2 m/s. Determine the magnitude and direction (expressed by the angle a relative to the *x*-axis) of the velocity and acceleration of the slider when *t* = 4 s.



Sample 14 (2/149)

The slotted arm *OA* forces the small pin to move in the fixed spiral guide defined by $r = K\theta$. Arm *OA* starts from rest at $\theta = \pi/4$ and has a constant counter clockwise angular acceleration $\ddot{\theta} = \alpha$. Determine the magnitude of the acceleration of the pin *P* when $\theta = 3\pi/4$.



Sample 15 (2/165)

For a limited range of motion, crank *CP* causes the slotted link *OA* to rotate. If β is increasing at the constant rate of 4 rad/s when $\beta = \pi/4$, determine the *r*- and θ-components of the acceleration of pin *P* for this position and specify the corresponding value of $\dot{r}$ and $\ddot{r}$.