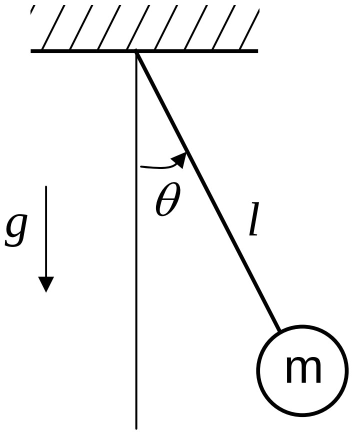
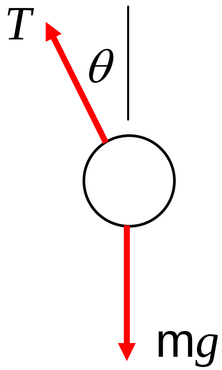
Linearization (introduction)

Consider the following example





$$\left[\sum M_O = J_O\ddot{\theta}\right]$$

$$-mgl\sin\theta = ml^2\ddot{\theta}$$

$$\boxed{\ddot{\theta} + \left(\frac{g}{l}\right)\sin\theta = 0}$$

Non-linear equation

- Difficult or cannot solve for the exact solution
- Superposition of solutions does not allow

Linear and non-linear equations

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=d$$

- a, b, c, d = constant, or $f(x)$ → Linear equation

- $a, b, c, d = f(y)$
- There are transcendental functions in the equation

→ Non-linear equation

Function that cannot be defined directly by algebraic formulas
Ex. Exponential, logarithmic, trigonometric functions

Linearization (1)

To obtain a linear model of non-linear mechanical systems or components based on Taylor series.

Non-linear model: $y = f(x)$

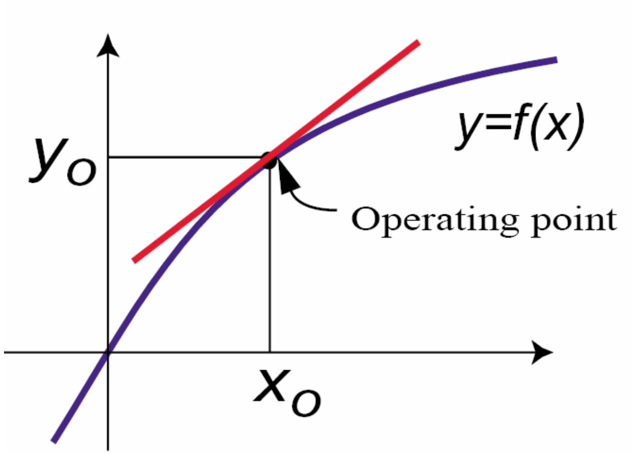
Taylor series

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0) + \ldots + \frac{(x - x_0)^n}{n!} f^{(n)}(x_0) + \ldots$$

Linearized model
(first-order approximation)

$$f(x) \cong f(x_0) + (x - x_0) f'(x_0)$$

This approximation is valid only for x near x_0 (operating point)



Linearization (2)

Example: $y = \sin(x)$

From $f(x) \cong f(x_0) + (x - x_0) f'(x_0)$

$$\sin(x) \cong \sin(x_0) + (x - x_0) \left. \frac{d(\sin(x))}{dx} \right|_{x=x_0}$$

$$\sin(x) \cong \sin(x_0) + (x - x_0) \cos(x_0)$$

$x_0 = 0 \quad \Longrightarrow \quad \sin(x) \cong 0 + (x - 0)(1) = x$ for x near 0

$x_0 = \dfrac{\pi}{2} \quad \Longrightarrow \quad \sin(x) \cong 1 + (x - \dfrac{\pi}{2})(0) = 1$ for x near $\pi/2$

$$\ddot{\theta} + \left(\frac{g}{l}\right) \sin\theta = 0 \quad \Longrightarrow \quad \ddot{\theta} + \left(\frac{g}{l}\right) \theta = 0$$

for small θ

$(\theta \to 0)$

Linearized EOM (1)

Given non-linear EOM: $M(x)\ddot{x} + C(x,\dot{x}) + K(x) = F(t)$

Procedures

(1) Select an "operating point" $\langle t_0, x_0, \dot{x}_0, \ddot{x}_0, F(t_0) \rangle$

(2) At operating point, we have

$$M(x_0)\ddot{x}_0 + C(x_0,\dot{x}_0) + K(x_0) = F(t_0)$$

(3) Near the operating point, we have

$$\left\langle \begin{array}{l} t = t_0 + \Delta t, \quad F(t) = F(t_0) + \Delta F(t) \\ x = x_0 + \Delta x, \quad \dot{x} = \dot{x}_0 + \Delta\dot{x}, \quad \ddot{x} = \ddot{x}_0 + \Delta\ddot{x} \end{array} \right\rangle$$

$$M(x_0 + \Delta x)[\ddot{x}_0 + \Delta\ddot{x}] + C(x_0 + \Delta x, \dot{x}_0 + \Delta\dot{x}) + K(x_0 + \Delta x) \\ = F(t_0) + \Delta F(t)$$

Linearized EOM (2)

$$M(x_0+\Delta x)[\ddot{x}_0+\Delta\ddot{x}]+C(x_0+\Delta x,\dot{x}_0+\Delta\dot{x})+K(x_0+\Delta x)$$
$$=F(t_0)+\Delta F(t)$$

(4) Using Taylor series for non-linear terms

$$M(x_0+\Delta x)\cong M(x_0)+\left.\frac{\partial M}{\partial x}\right|_{x_0}\Delta x$$

$$C(x_0+\Delta x,\dot{x}_0+\Delta\dot{x})\cong C(x_0,\dot{x}_0)+\left.\frac{\partial C}{\partial \dot{x}}\right|_{x_0,\dot{x}_0}\Delta\dot{x}+\left.\frac{\partial C}{\partial x}\right|_{x_0,\dot{x}_0}\Delta x$$

$$K(x_0+\Delta x)\cong K(x_0)+\left.\frac{\partial K}{\partial x}\right|_{x_0}\Delta x$$

Linearized EOM (3)

(5) Substituting (4) into (3), using relation in (2) and neglecting higher-order terms,

$$\tilde{M}\Delta\ddot{x}+\tilde{C}\Delta\dot{x}+\tilde{K}\Delta x=\Delta F(t)$$

Where $\tilde{M} = M(x_0)$

$$\tilde{C} = \left.\frac{\partial C}{\partial \dot{x}}\right|_{x_0,\dot{x}_0}$$

$$\tilde{K} = \left.\frac{\partial M}{\partial x}\right|_{x_0}\Delta\ddot{x}_0 + \left.\frac{\partial C}{\partial x}\right|_{x_0,\dot{x}_0} + \left.\frac{\partial K}{\partial x}\right|_{x_0}$$

This “linearized EOM” is valid only near the operating point x_0

⇨ $\Delta x, \Delta\dot{x}, \Delta\ddot{x}, \Delta F(t)$ are small

Example (1)

Derive the EOM of the system in the figure. You may have to make some approximation of the cosine. Assume the bearings provide a viscous damping force only in the vertical direction. [Inman/1.46]

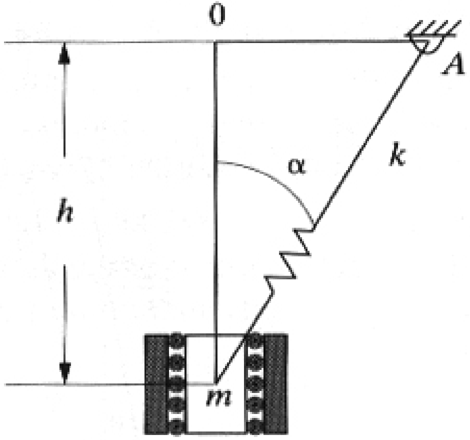


Figure P1.46

Example (2)

Derive the EOM of an airplane’s steering-gear mechanism for the nose wheel of its landing gear. The mechanism is modeled as the single-degree-of-freedom system illustrated in the figure. [Inman/1.49]

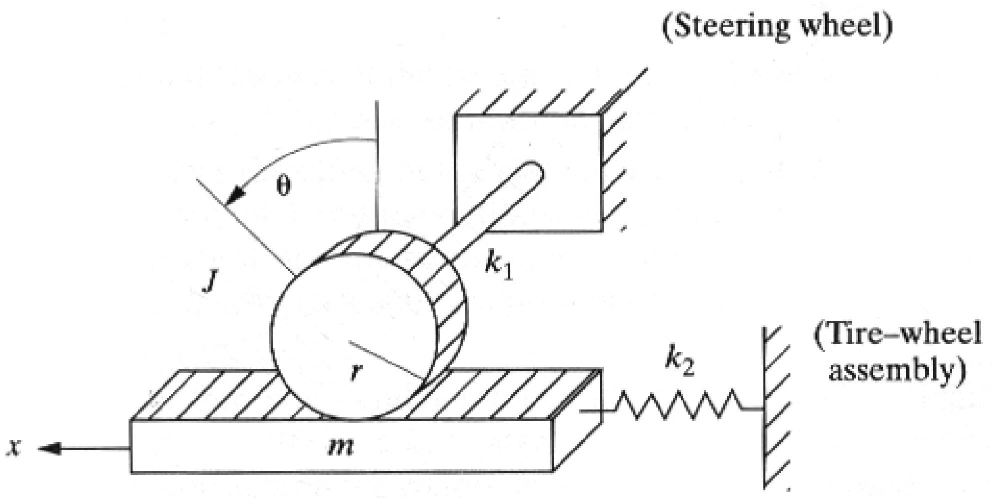
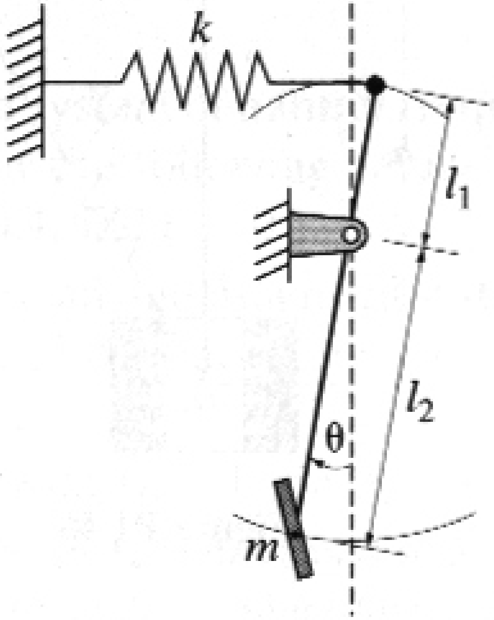


Figure P1.49 Single-degree-of-freedom model of a steering mechanism.

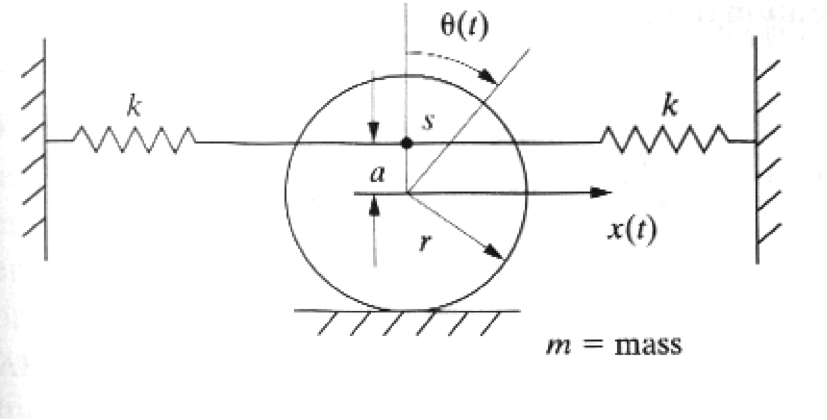
Example (3)

A control pedal of an aircraft can be modeled as the single-degree-of-freedom system shown in the figure. Consider the level as a massless shaft and the pedal as a lumped mass at the end of the shaft. Determine the EOM in θ. Assume the spring to be unstretched at $\theta = 0$. [Inman/1.50]



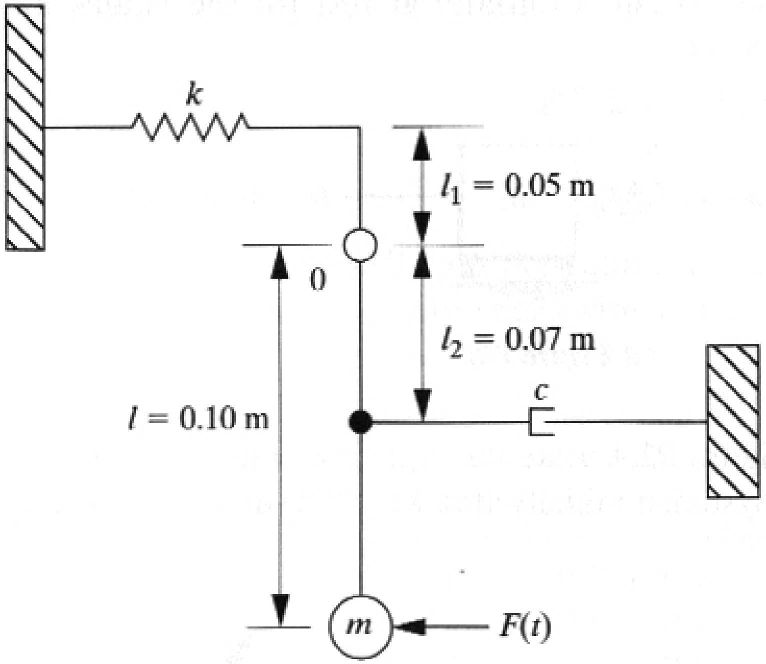
Example (4)

Consider the disk of the figure connected to two springs. Derive EOM for small angle $\theta(t)$. [Inman/1.82]



Example (5)

Consider the pendulum mechanism shown in the figure, which is pivoted at point O. Derive EOM for small angles. Assume that the masses of the rod, spring, and damper are negligible. [Inman/2.16]



Example (6)

A foot pedal for a musical instrument is modeled by the sketch in the figure. Derive EOM of the system. Also, use the small angle approximation. [Inman/2.25]

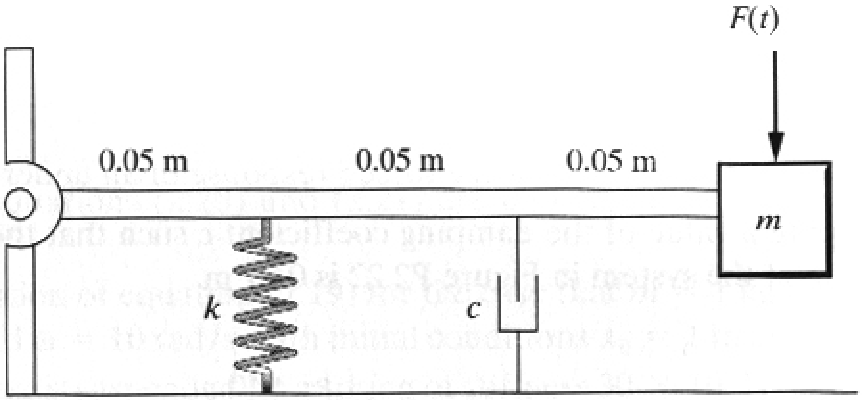


Figure P2.25