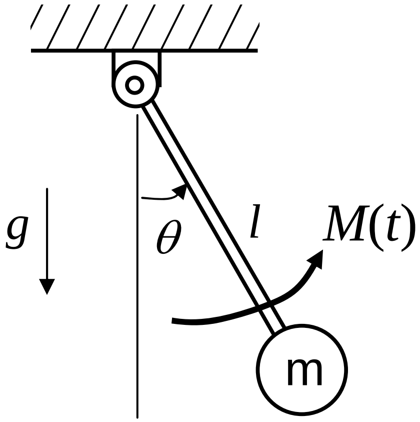
Equilibrium and stability condition

Equilibrium A "configuration" at which a mechanical system stays at rest as long as there is no external disturbances (excitations).

Determination of equilibrium configurations

Example



EOM: $ml^2\ddot{\theta} + mgl\sin\theta = M(t)$

At equilibrium configuration $\theta = \theta_0$

System stays at rest $\ddot{\theta} = \dot{\theta} = 0$

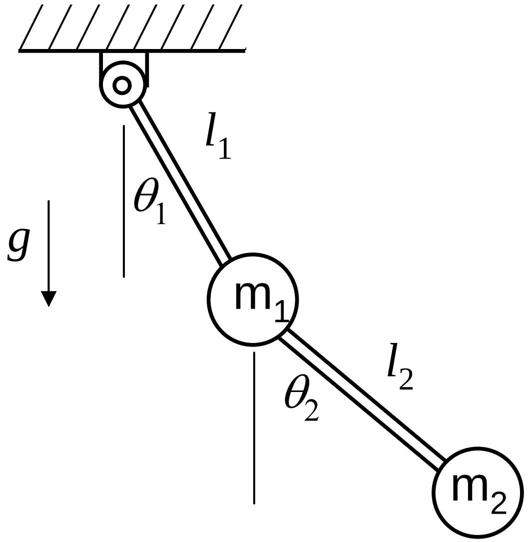
No external force and moment $M(t) = 0$

$$\cancel{ml^2\ddot{\theta}}^{\,0} + mgl\sin\theta_0 = \cancel{M(t)}^{\,0}$$

$$\sin\theta_0 = 0 \quad \Longrightarrow \quad \theta_0 = \pm n\pi \qquad n = 0,1,2\ldots$$

This pendulum has 2 equilibrium configs. at $\theta_0 = 0$ and π

Example: Equilibrium config.



Given EOM

$$m_1 l_1^2 \ddot{\theta}_1 + m_1 g l_1 \sin\theta_1 - m_2 g \sin\theta_2 (l_1 \cos\theta_1) + m_2 g \cos\theta_2 (l_1 \sin\theta_1) = 0$$

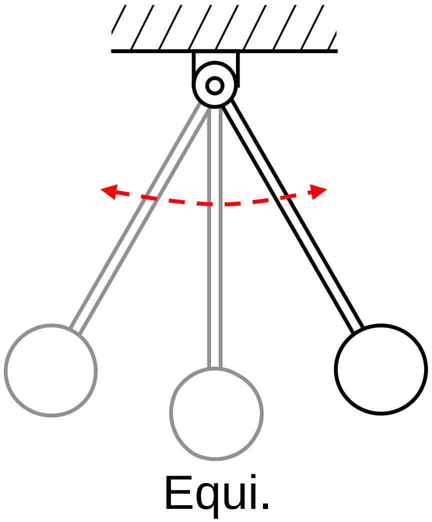
$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_2 (l_1 \ddot{\theta}_1) + m_2 g l_2 \sin\theta_2 = 0$$

Determine equilibrium configuration.

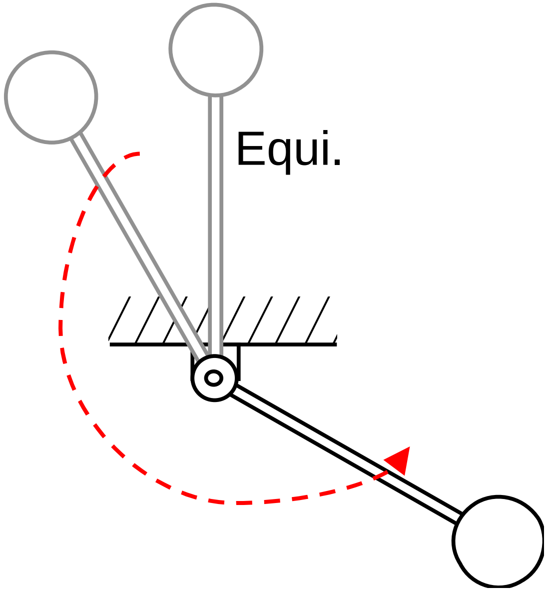
(assume small angle θ_1 and θ_2, and force in rod $l_2 \cong m_2 g$)

Stability condition

A mechanical system disturbed from a “stable” equilibrium configuration oscillates about and/or returns to the same equilibrium configuration.



Stable at $\theta = 0$



Unstable at $\theta = \pi$

Determine stability condition (1)

Consider EOM

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0$$

If m, c, $k > 0$ The solution is,

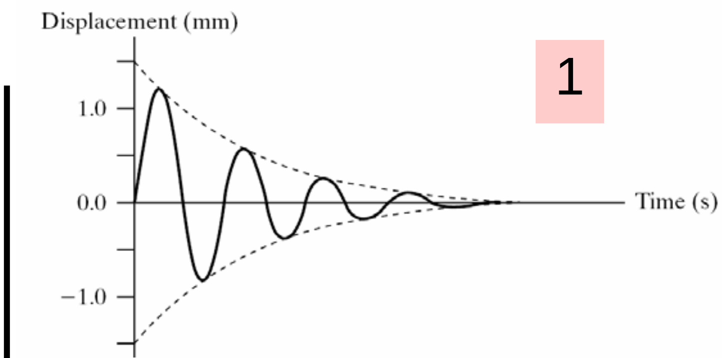
$$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

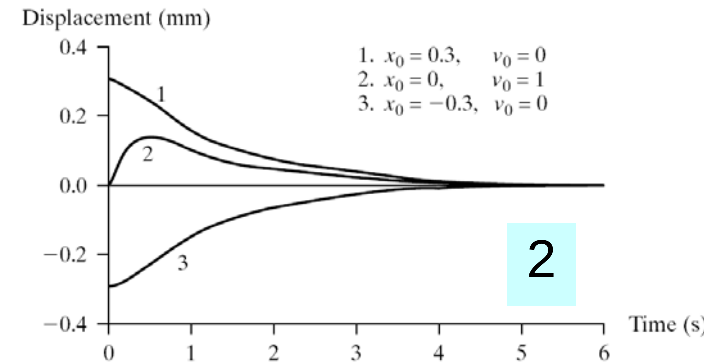
or

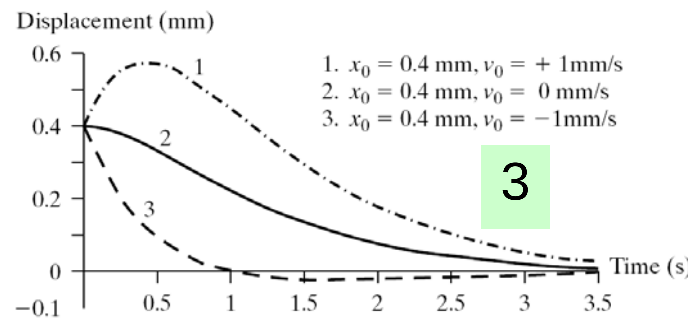
$$x(t) = e^{-\zeta\omega_n t}\left(a_1 e^{-\omega_n\sqrt{\zeta^2-1}t} + a_2 e^{+\omega_n\sqrt{\zeta^2-1}t}\right)$$

or

$$x(t) = (a_1 + a_2 t)e^{-\omega_n t}$$







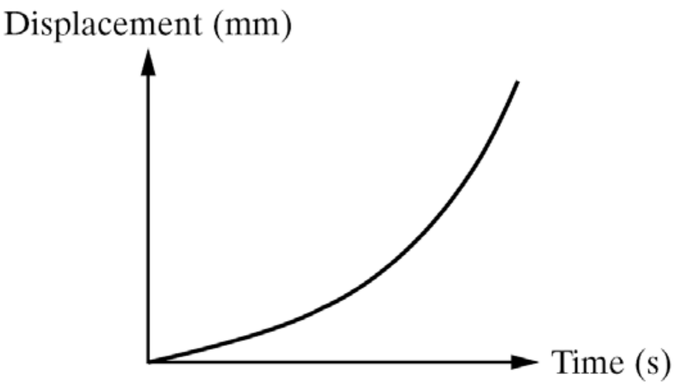
Determine stability condition (2)

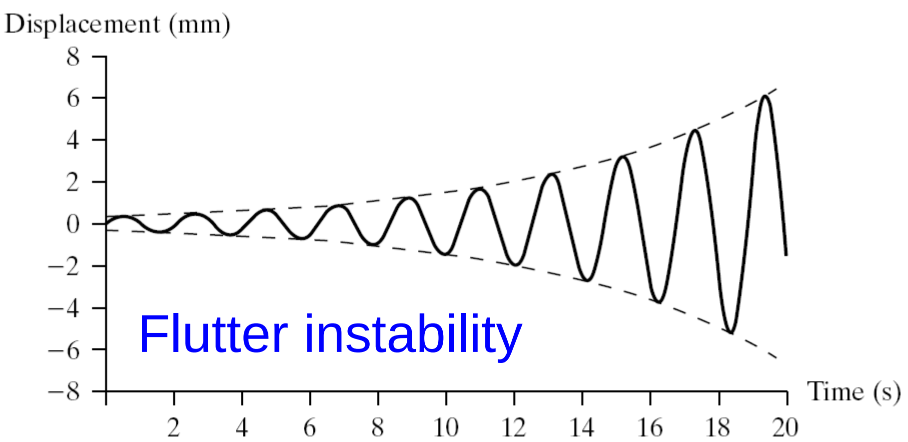
$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0$$

If m, c, $k > 0$ $\quad$ $x(t)$ approaches zero as t becomes large

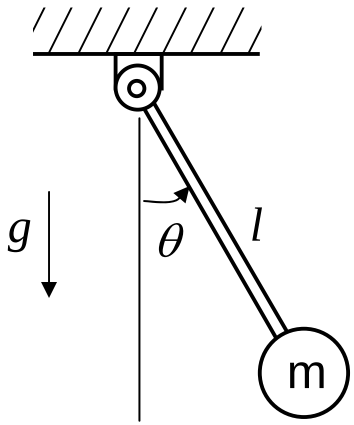
Asymptotically stable

If c, $k < 0$ but $m > 0$ $\quad$ Unstable, divergent





Example: pendulum



EOM: $ml^2\ddot{\theta} + mgl\sin\theta = 0$

This pendulum has 2 equilibrium configs.
at $\theta_0 = 0$ and π

Consider at $\theta_0 = \pi$, linearize term $\sin\theta$

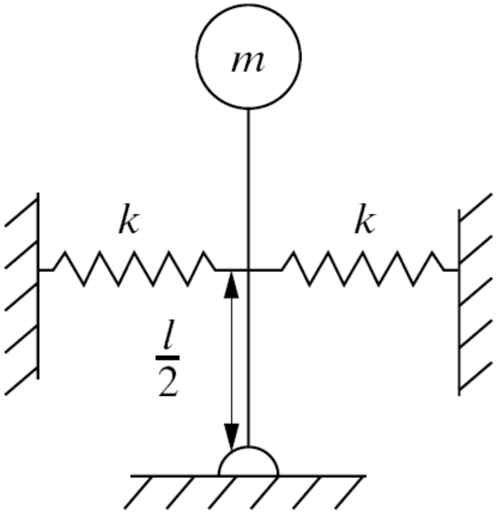
$$\sin(\theta) \cong \sin(\theta_0) + (\theta - \theta_0)\cos(\theta_0)$$

$$\sin(\theta) \cong 0 + (\theta - \pi)(-1) = (\pi - \theta)$$

Sub. $\sin\theta$ in EOM, yield $ml^2\ddot{\theta} - mgl\theta = -mgl\pi$

<0, unstable

Example: Inverted pendulum



Conclusion modeling

Assumptions

1. **Neglect small effects**

   Reduces number and complexity of differential equations

2. **Lumped characteristics**

   Leads to ordinary (rather than partial) differential equations

3. **Linearity**

   Makes equations linear, allow superposition of solutions

4. **Constant parameters**

   Leads to constant coefficients in differential equations

5. **Neglecting uncertainly and noise**

   Avoids statistical treatment

Home work

Consider the disk of the figure connected to two springs.
Derive EOM for small angle $\theta(t)$. [Inman/1.82]