Free vibration (Introduction)

A system is said to undergo free vibration when it oscillates only under an initial disturbance with no external forces after the initial disturbance.

Examples

- The oscillations of the pendulum of a clock
- The vertical oscillatory motion felt by a bicyclist after hitting a road bump
- The motion of a child on a swing under an initial push

Free vibration (EOM)

EOM of a single degree of freedom system

$$\underbrace{m\ddot{x}(t)+c\dot{x}(t)+kx(t)}_{\text{System's characteristics}} = \underbrace{F(t)}_{\text{External forces}}$$

Free vibration: External forces $= 0$

$$\Rightarrow \boxed{m\ddot{x}(t)+c\dot{x}(t)+kx(t)=0}$$

The response of the system $x(t)$ can be known from the solution of the EOM

Review differential equation (1)

Consider the differential equation shown below

$$a_1 \frac{d^2 y}{dx^2} + a_2 \frac{dy}{dx} + a_3 y = 0$$

Auxiliary equation: $a_1 r^2 + a_2 r + a_3 = 0$

Roots of the auxiliary equation: r_1, r_2

Review differential equation (2)

Case 1: r_1 and r_2 are real numbers and $r_1 \neq r_2$

Solution of differential equation:

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

Case 2: r_1 and r_2 are real numbers and $r_1 = r_2$

Solution of differential equation:

$$y = (C_1 + C_2 x) e^{r_1 x}$$

Review differential equation (3)

Case 3: r_1 and r_2 are complex numbers $r_1 = a+bi$, $r_2 = a-bi$

Solution of differential equation:

$$y = C_1 e^{(a+bi)x} + C_2 e^{(a-bi)x}$$

From Euler identity: $e^{i\theta} = \cos\theta + i\sin\theta$

$$y = e^{ax}(A_1\cos(bx) + A_2\sin(bx))$$

or

$$y = Ae^{ax}\sin(bx+\phi)$$

Free vibration system (undamped)

EOM of the undamped free vibration system

$$m\ddot{x}(t) + kx(t) = 0$$

Auxiliary equation: $mr^2 + k = 0$

Roots of the auxiliary equation: $\pm i\left(\sqrt{k/m}\right)$

(m and k are always positive, the roots of the aux. eq. are complex numbers)

Solution of differential equation:

$$x = A_1 \cos(\sqrt{\frac{k}{m}}t) + A_2 \sin(\sqrt{\frac{k}{m}}t))$$

or $$x = A\sin(\sqrt{\frac{k}{m}}t + \phi)$$

(A_1 and A_2 or A and ϕ are obtained from initial conditions)

Response of the free vibration system

Consider the response in the term

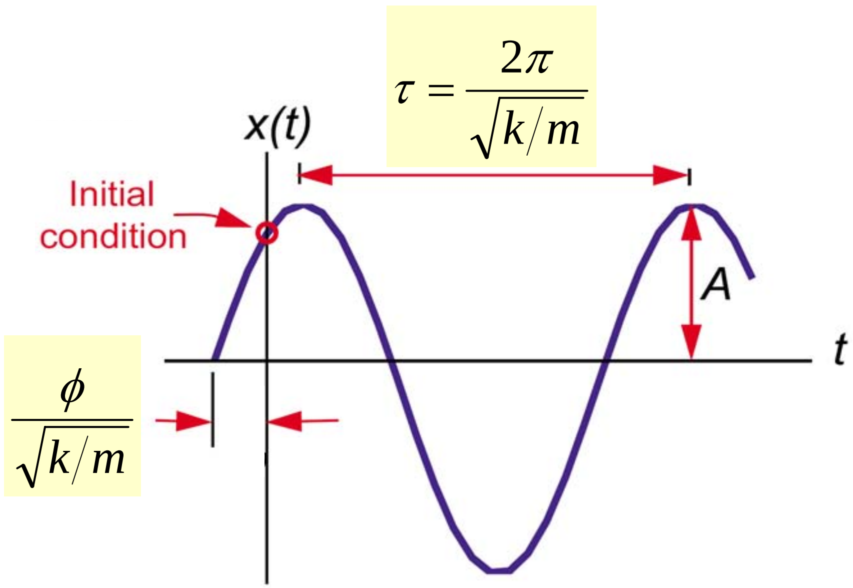
$$x = A\sin\left(\sqrt{\frac{k}{m}}\,t + \phi\right)$$

$\sqrt{k/m}$ is call “natural frequency”

$$\omega_n = \sqrt{k/m}$$

and

$$\omega_n = 2\pi f_n = 2\pi/\tau$$



- Free vibration only occurs at a certain frequency ω_n.
- Response is sinusoidal and not decaying (undamped).

Initial condition (1)

Given EOM: $m\ddot{x}(t) + kx(t) = 0$

Initial condition: $x(0) = x_0$, $\dot{x}(0) = v_0$

From given EOM $\ddot{x}(t) + \dfrac{k}{m}x(t) = 0 \Rightarrow \ddot{x}(t) + \omega_n^2 x(t) = 0$

Response of the system is $x(t) = A\sin(\omega_n t + \phi)$

$$\dot{x}(t) = A\omega_n \cos(\omega_n t + \phi)$$

Initial condition: $x(0) = x_0$, $\dot{x}(0) = v_0$ $\Rightarrow$

$$x_0 = A\sin(\phi) \qquad (1)$$

$$v_0 = A\omega_n \cos(\phi) \qquad (2)$$

Initial condition (2)

Given EOM: $m\ddot{x}(t) + kx(t) = 0$

Initial condition: $x(0) = x_0$, $\dot{x}(0) = v_0$

Solving equations (1) and (2) yield,

$$A = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n} \quad \text{and} \quad \phi = \tan^{-1}\frac{\omega_n x_0}{v_0}$$

Therefore the response is

$$x(t) = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n} \sin(\omega_n t + \tan^{-1}\frac{\omega_n x_0}{v_0})$$

Relationship between *x*, *v*, and *a*.

Displacement

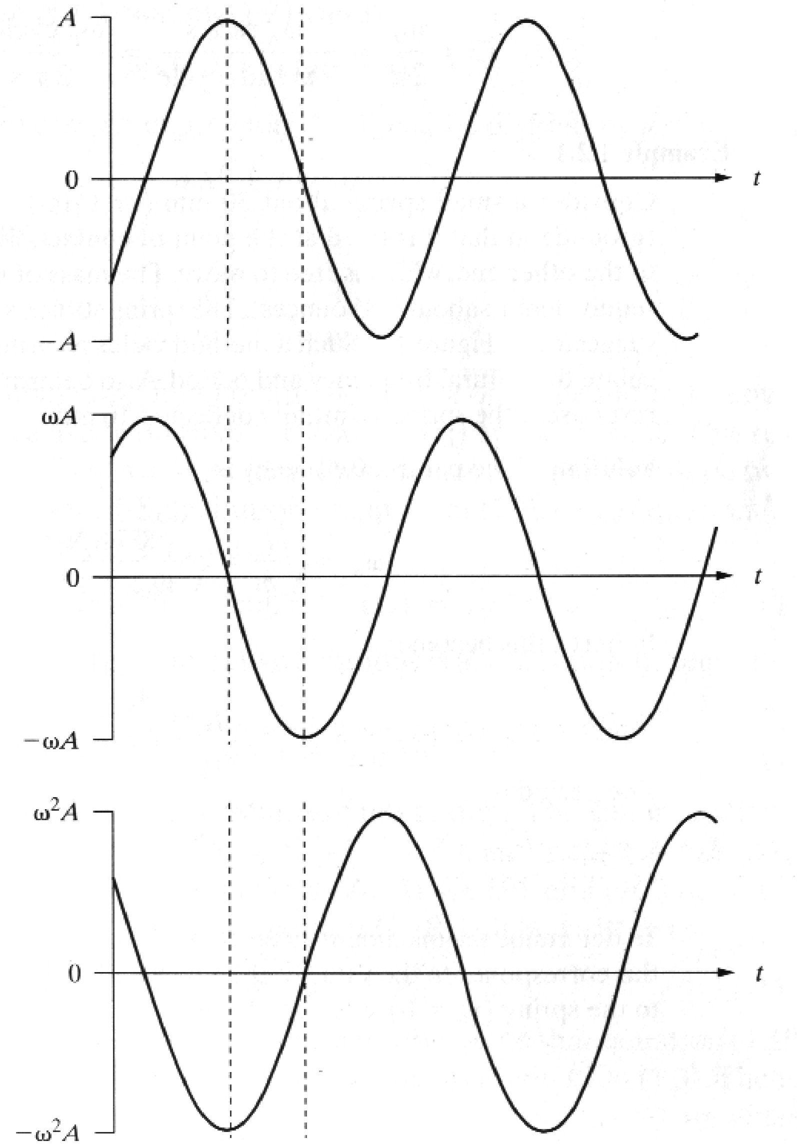
$$x(t) = A\sin(\omega_n t + \phi)$$

Velocity

$$\dot{x}(t) = \omega_n A\cos(\omega_n t + \phi)$$

Acceleration

$$\ddot{x}(t) = -\omega_n^2 A\sin(\omega_n t + \phi)$$



Example: (m, k, ω_n)

A vehicle wheel, tire, and suspension assembly can be modeled crudely as a single-degree-of-freedom spring-mass system. The mass of the assembly is measured to be about 300 kilograms (kg). Its frequency of oscillation is observed to be 10 rad/s. What is the approximate stiffness of the tire, wheel, and suspension assembly? [Inman ex1.1.2]

Example: Young's Modulus from f_n

A simply supported beam of square cross section 5mm$\times$5mm and length 1m, carrying a mass of 2.3kg at the middle, is found to have a natural frequency of transverse vibration of 30 rad/s. Determine the Young's modulus of elasticity of the beam.