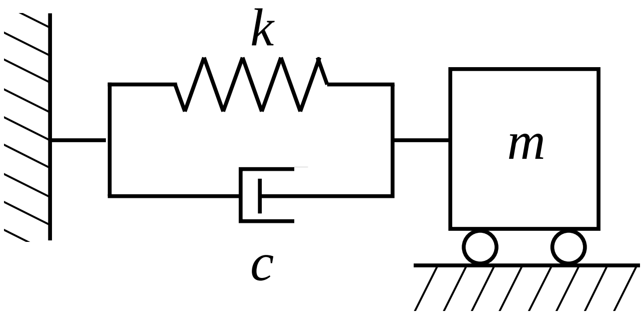
Viscously damped free vibration (1)



EOM of viscously damped free vibration system:

$$m\ddot{x} + c\dot{x} + kx = 0$$

Auxiliary equation: $mr^2 + cr + k = 0$

Roots of the auxiliary equation: $r_{1,2} = \dfrac{-c \pm \sqrt{c^2 - 4mk}}{2m}$

$c^2 - 4mk > 0$ ⇨ $r_1,\ r_2$ (different real numbers)

$c^2 - 4mk = 0$ ⇨ $r = -c/2m$

$c^2 - 4mk < 0$ ⇨ $r_1,\ r_2$ (complex numbers)

Viscously damped free vibration (2)

$\sqrt{c^2 - 4mk} = 0$ ⇨ c = "the critical damping coefficient, c_{cr}"

$$c_{cr} = 2\sqrt{mk} = 2m\omega_n$$

Furthermore, nondimensional number ζ called the "damping ratio" defined by

$$\zeta = \frac{c}{c_{cr}} = \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{km}}$$

Viscously damped free vibration (3)

With these definitions, EOM becomes

$$m\ddot{x} + c\dot{x} + kx = 0 \quad \Rightarrow \quad \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

Roots of auxiliary equation become

$$r_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} \quad \Rightarrow \quad r_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$\zeta^2 - 1 < 0 \Rightarrow 0 < \zeta < 1 \Rightarrow$ Underdamped motion

$\zeta^2 - 1 > 0 \Rightarrow \zeta > 1 \Rightarrow$ Overdamped motion

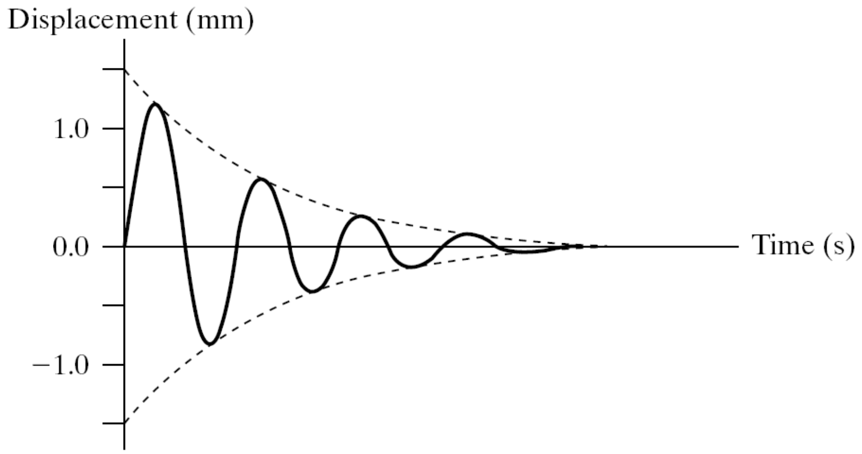
$\zeta^2 - 1 = 0 \Rightarrow \zeta = 1 \Rightarrow$ Critically damped motion

Underdamped motion

$$\zeta^2 - 1 < 0 \Rightarrow 0 < \zeta < 1$$

Roots of auxiliary equation become $r_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}\,j$

Solution of diff. equation: $$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$



Where $\omega_d = \omega_n\sqrt{1-\zeta^2}$

A and ϕ are obtained from initial condition.

Overdamped motion

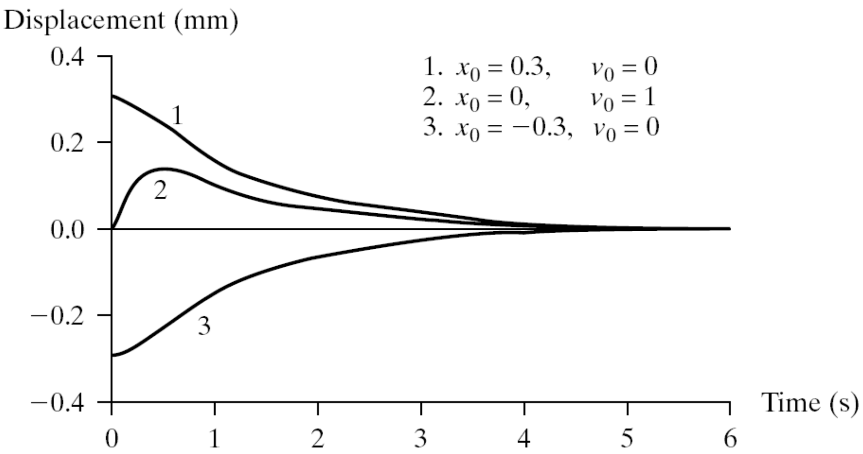
$$\zeta^2 - 1 > 0 \quad \Rightarrow \quad \zeta > 1$$

Roots of auxiliary equation become $r_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

Solution of diff. equation:

$$x(t) = e^{-\zeta\omega_n t}\left(a_1 e^{-\omega_n\sqrt{\zeta^2-1}t} + a_2 e^{+\omega_n\sqrt{\zeta^2-1}t}\right)$$

a_1 and a_2 are obtained from initial condition.



Critically damped motion

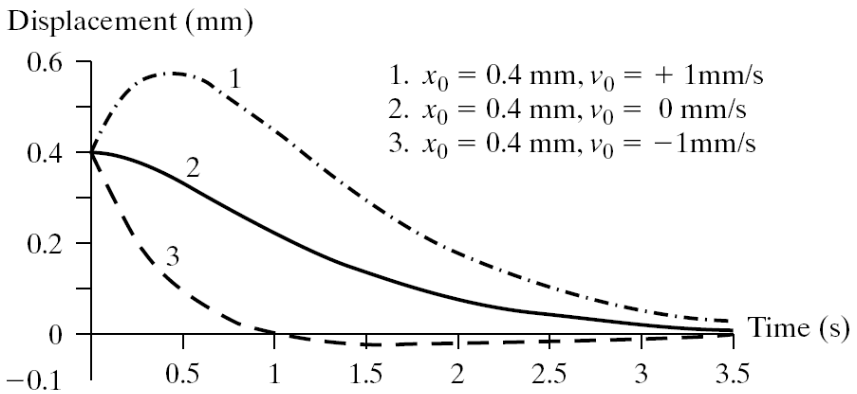
$$\zeta^2 - 1 = 0 \Rightarrow \zeta = 1$$

Roots of auxiliary equation become $r_{1,2} = -\zeta\omega_n = -\omega_n$

Solution of diff. equation:

$$x(t) = (a_1 + a_2 t)e^{-\omega_n t}$$

a_1 and a_2 are obtained from initial condition.



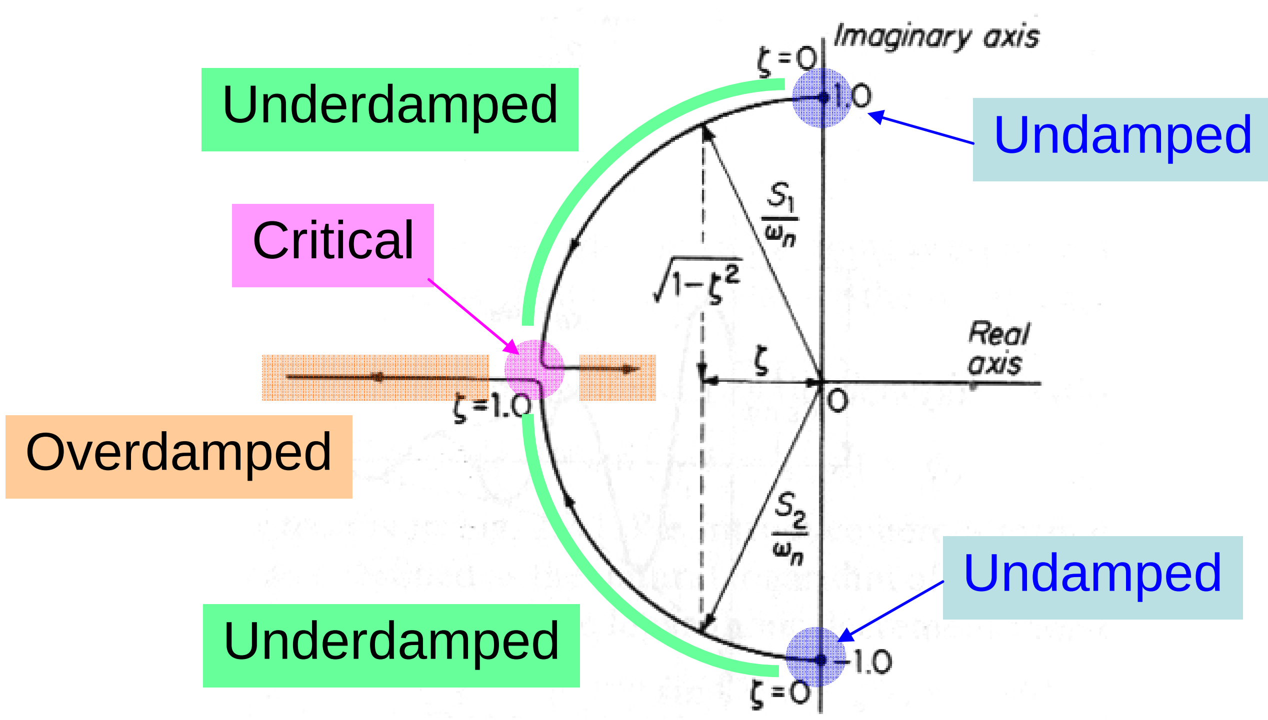
Note:

- Smallest value of damping rate that yield aperiodic motion.
- The value of damping ratio that provides fastest return to zero without oscillation.

Characteristic roots on complex plane

Roots of auxiliary equation $r_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}\,j$

$$\frac{r_{1,2}}{\omega_n} = -\zeta \pm \sqrt{1-\zeta^2}\,j \qquad 0<\zeta<1$$



Example (1)

A Spring-mass-damper system has mass of 100 kg, stiffness of 3000 N/m and damping coefficient of 300 kg/s. Calculate the undamped natural frequency, the damping ratio and the damped natural frequency. Does the solution oscillate? This system is given a zero initial velocity and an initial displacement of 0.1 m. Calculate the vibration response. [inman 1.40, 1.42]

Example (2)

A Spring-mass-damper system has mass of 150 kg, stiffness of 1500 N/m and damping coefficient of 200 kg/s. Calculate the undamped natural frequency, the damping ratio and the damped natural frequency. Is the system overdamped, underdamped or critically damped? Does the solution oscillate?

This system is given an initial velocity of 10 mm/s and an initial displacement of -5 mm. Calculate the vibration response. [inman 1.41, 1.43]