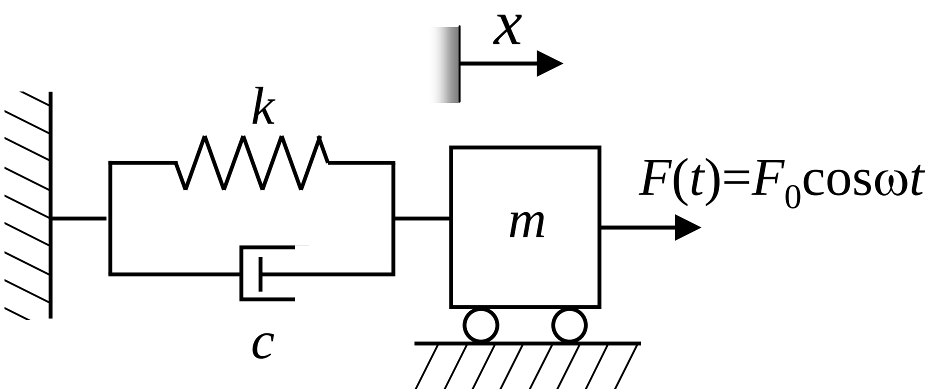
Harmonic excitation (damped)



EOM:

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t$$

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = f_0 \cos \omega t$$

The response $x(t)$ (solution) can be separated into 2 part;

1. Homogeneous solution $x_h(t)$ → $\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = 0$
2. Particular solution $x_p(t)$ → $\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = f_0 \cos \omega t$

$$x(t) = x_h(t) + x_p(t)$$

Homogeneous solution

Homogeneous solution $x_h(t)$ ➡ Same as free vibration

Under damped $0 < \zeta < 1$

$$x_h(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$ where $\omega_d = \omega_n\sqrt{1-\zeta^2}$

Over damped $1 < \zeta$

$$x_h(t) = A_1 e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + A_2 e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}$$

Critically damped $\zeta = 1$

$$x_h(t) = (A_1 + A_2 t)e^{-\omega_n t}$$

Particular solution

From EOM $\ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2 x = f_0\cos\omega t$

The particular solution $x_p(t)$ can be written in the form:

$$x_p(t) = X\cos(\omega t-\theta)$$

or

$$x_p(t) = A_s\cos(\omega t)+B_s\sin(\omega t)$$

Substitution $x_p(t)$ into EOM, the coefficients X and θ (A_s and B_s) can be determined. Then the particular solution is

$$x_p(t) = \underbrace{\frac{f_0}{\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}}}_{X}\cos(\omega t-\tan^{-1}\underbrace{\frac{2\zeta\omega_n\omega}{\omega_n^2-\omega^2})}_{\theta}$$

Response of harmonic excitation (1)

Response of harmonic excitation $x(t) = \underline{x_h(t)} + x_p(t)$

Depend on ζ

Ex Underdamped

$$x(t) = \underbrace{Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)}_{x_h(t)} + \underbrace{X\cos(\omega t - \theta)}_{x_p(t)}$$

$x_h(t)$

Decrease with time

Transient response

$x_p(t)$

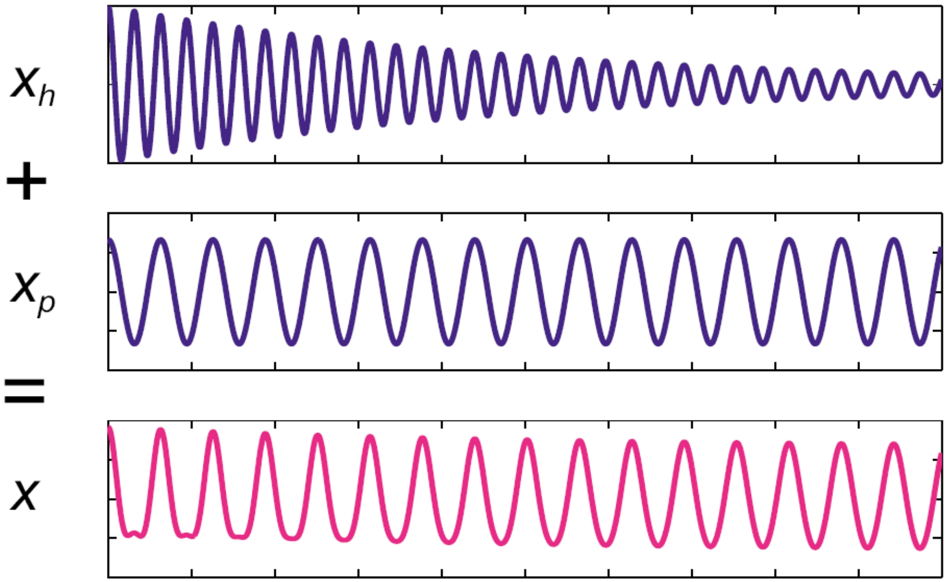
Amplitude X is constant

Steady-state response

Response of harmonic excitation (2)

Ex Underdamped

$$x(t) = \underbrace{Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)}_{x_h(t)} + \underbrace{X\cos(\omega t - \theta)}_{x_p(t)}$$



As $t \rightarrow \infty$, transient response dies out and total response

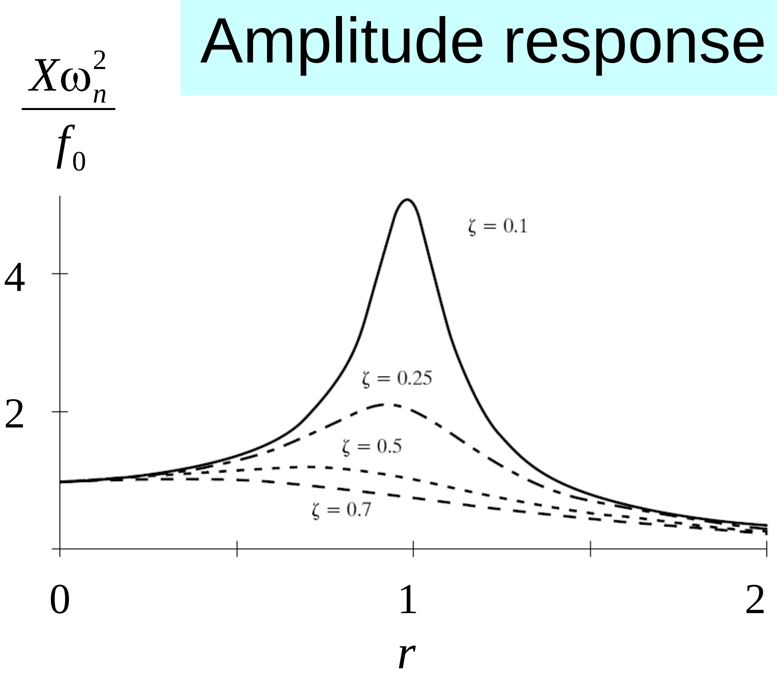
$$x(t) \rightarrow x_p(t)$$

Steady-state response (1)

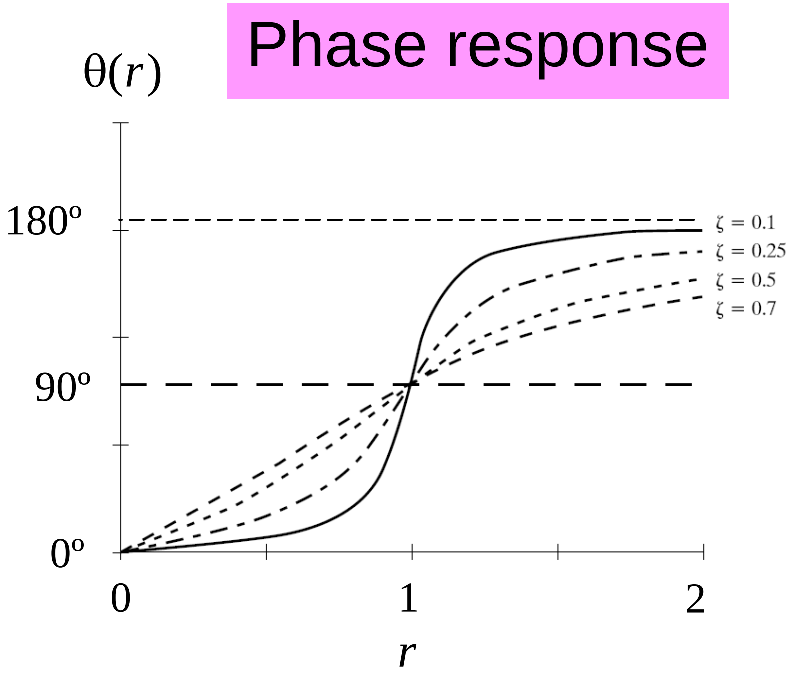
From steady-state response $x_p(t) = \dfrac{f_0}{\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}}\cos(\omega t - \tan^{-1}\dfrac{2\zeta\omega_n\omega}{\omega_n^2-\omega^2})$

$\Rightarrow$ $\dfrac{Xk}{F_0} = \dfrac{X\omega_n^2}{f_0} = \dfrac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$, $\theta = \tan^{-1}\dfrac{2\zeta r}{1-r^2}$ where $r = \dfrac{\omega}{\omega_n}$

Amplitude response



Phase response



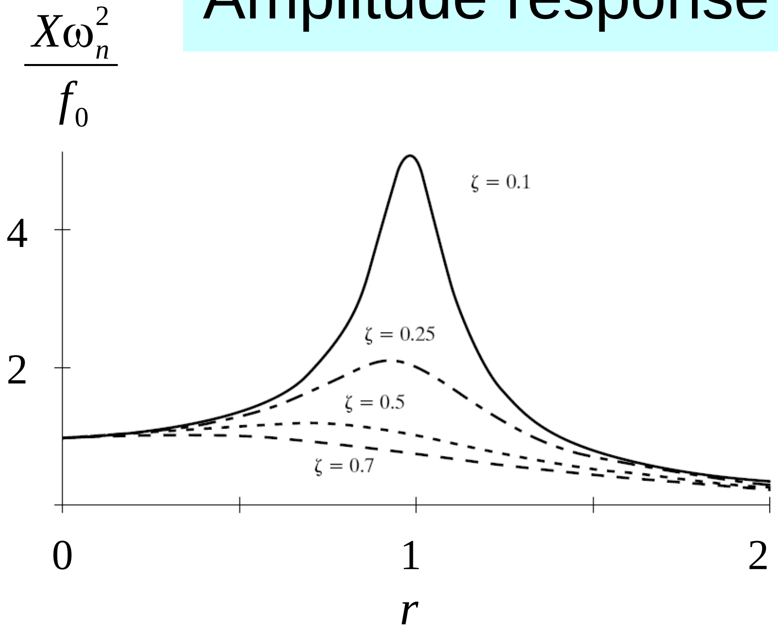
Steady-state response (2)

From steady-state response $x_p(t) = \dfrac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \cos(\omega t - \tan^{-1} \dfrac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2})$

$$\Rightarrow \quad \frac{Xk}{F_0} = \frac{X\omega_n^2}{f_0} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

where $r = \dfrac{\omega}{\omega_n}$

Amplitude response



Maximum amplitude occurs when

$$\frac{d}{dr}\left(\frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}\right) = 0$$

$$\Downarrow$$

$r = \sqrt{1 - 2\zeta^2}$ or $\omega = \omega_n \sqrt{1 - 2\zeta^2}$

$$\left(\frac{Xk}{F_0}\right)_{\max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

Frequency response method (1)

Euler's formula $e^{j\omega t} = \cos\omega t + j\sin\omega t$

EOM $m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F_0 \cos\omega t$

EOM (complex form) $m\ddot{z}(t) + c\dot{z}(t) + kz(t) = F_0 e^{j\omega t}$

Real part of the complex solution corresponds to the solution of EOM.

⇒ $x_p(t) = \mathrm{Re}(z)$ where $z = Ze^{j\omega t}$

↓

Z is a complex-valued constant

Substituting z into EOM (complex form).

$$(-\omega^2 m + jc\omega + k)Ze^{j\omega t} = F_0 e^{j\omega t}$$

Frequency response method (2)

$$(-\omega^2 m + jc\omega + k)Ze^{j\omega t} = F_0 e^{j\omega t}$$

$$Z = \frac{1}{(k - m\omega^2) + (c\omega)j} \cdot F_0 = H(j\omega)F_0$$

$H(j\omega) =$ (complex) frequency response function

$$Z = \frac{F_0}{[(k - m\omega^2)^2 + (c\omega)^2]^{1/2}} \cdot e^{-j\theta}, \quad \text{where} \quad \theta = \tan^{-1}\frac{c\omega}{(k - m\omega^2)}$$

$$z = \frac{F_0}{[(k - m\omega^2)^2 + (c\omega)^2]^{1/2}} \cdot e^{j(\omega t - \theta)}$$

$$x_p(t) = \mathrm{Re}(z) \quad \Rightarrow \quad x_p(t) = \frac{F_0}{[(k - m\omega^2)^2 + (c\omega)^2]^{1/2}} \cdot \cos(\omega t - \theta)$$

Frequency response method (3)

$$x_p(t) = \frac{F_0}{[(k-m\omega^2)^2+(c\omega)^2]^{1/2}} \cdot \cos(\omega t - \theta)\ ; \quad \theta = \tan^{-1}\frac{c\omega}{(k-m\omega^2)}$$

or

$$x_p(t) = \frac{f_0}{\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}} \cos(\omega t - \theta)\ ; \quad \theta = \tan^{-1}\frac{2\zeta\omega_n\omega}{\omega_n^2-\omega^2}$$

$x_p(t)$ in the form of frequency response function

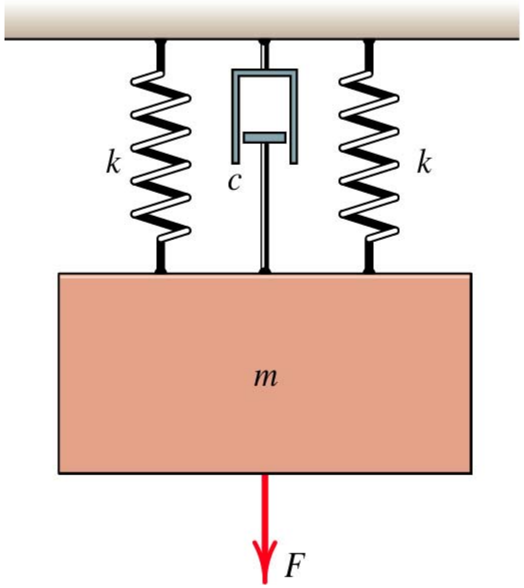
$$Z = H(j\omega)F_0 = |H(j\omega)|F_0 \cdot e^{-j\theta}$$

$$z = |H(j\omega)|F_0 \cdot e^{j(\omega t-\theta)}$$

$$x_p(t) = \mathrm{Re}(z) \quad \Rightarrow \quad x_p(t) = |H(j\omega)|F_0 \cdot \cos(\omega t - \theta)$$

Example 1

The block of mass m = 45 kg is suspended by two springs each of stiffness k = 3 kN/m and is acted upon by the force $F = 350\cos(15t)$ N where t is the time in seconds. Determine the amplitude X of the steady-state motion if the viscous damping coefficient c is (a) 0 and (b) 900 Ns/m. [J. L. Meriam & L. G. Kraige 8/52]



Example 2

For a vibrating system, m = 10 kg, k = 2500 N/m, and c = 45 Ns/m. A harmonic force of amplitude 180 N and frequency 3.5 Hz acts on the mass. If the initial displacement and velocity of the mass are 15 mm and 5m/s, find the complete solution representing the motion of the mass.
[Singiresu S. Rao, Mechanical Vibrations 4th edition in SI units 3/33]

Example 3

A machine part of mass 1.95 kg vibrates in a viscous medium. Determine the damping coefficient when a harmonic exciting force of 24.46 N results in a resonant amplitude of 1.27 cm with a period of 0.2 s. [William T. Thomson & Marie Dillon Dahleh, Theory of Vibration with Applications 5th edition 3/1]

Example 4

A weight attached to a spring of stiffness 525 N/m has a viscous damping device. When the weight is displaced and released, the period of vibration is 1.80 s, and the ratio of consecutive amplitudes is 4.2 to 1.0. Determine the amplitude and phase when a force F = 2cos(3t) acts on the system. [William T. Thomson & Marie Dillon Dahleh, Theory of Vibration with Applications 5th edition 3/3]

Example 5

A spring-mass is excited by a force $F_0\cos(\omega t)$. At resonance, the amplitude is measured to be 0.58 cm. At 0.8 resonant frequency, the amplitude is measured to be 0.46 cm. Determine the damping factor ζ of the system. [William T. Thomson & Marie Dillon Dahleh, Theory of Vibration with Applications 5th edition 3/5]