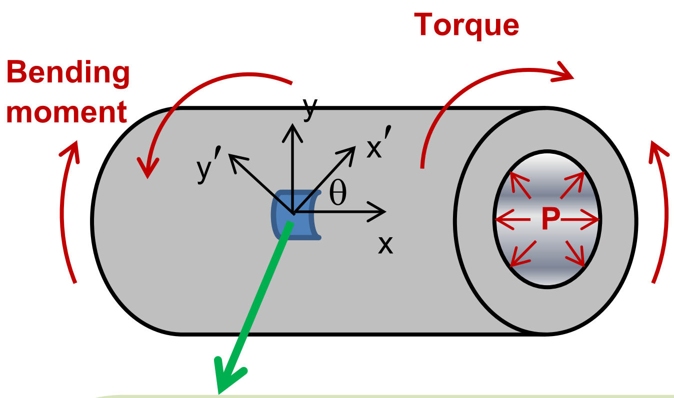
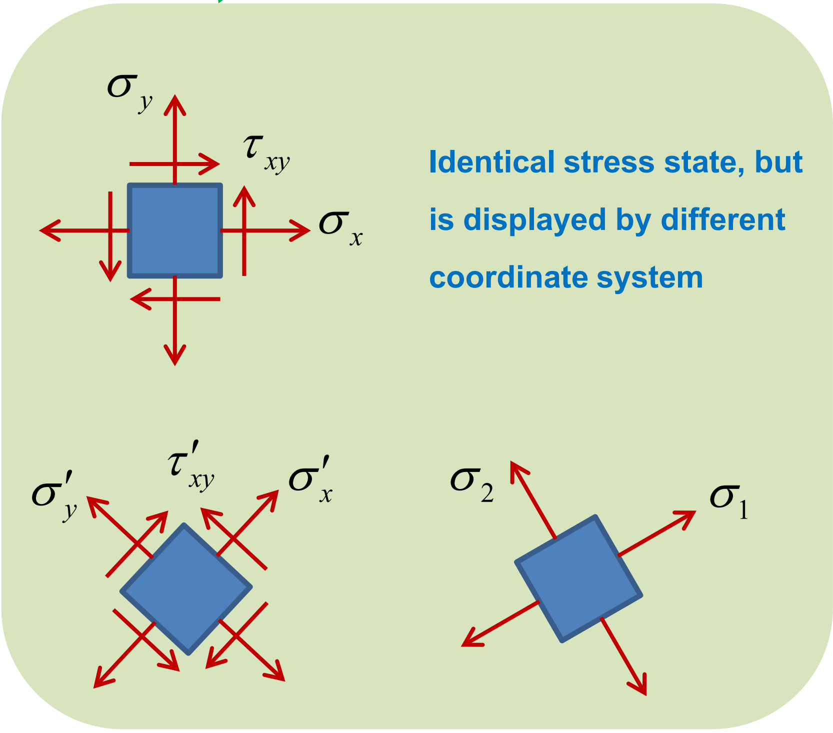
Failure Theories

- Review stress transformation
- Failure theories for ductile materials
- Maximum-Shear-Stress Theory
- Distortion-Energy Theory
- Coulomb-Mohr Theory
- Failure theories for brittle materials
- Maximum-Normal-Stress Theory
- Modifications of the Mohr Theory

2103320 Des Mach Elem
Mech. Eng. Department
Chulalongkorn University

Stress transformation





- At a point, there is only on stress state (σ_x, σ_y, τ_{xy})
- Using different coordinate, identical stress state can be written by different σ_x, σ_y, τ_{xy} (ex σ'_x, σ'_y, τ'_{xy})
- At a proper coordinate, $\tau'_{xy} = 0$ and only σ_1, σ_2 exist. This coordinate is called "Principal coordinate"

$$\sigma'_x = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma'_y = \frac{(\sigma_x + \sigma_y)}{2} - \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

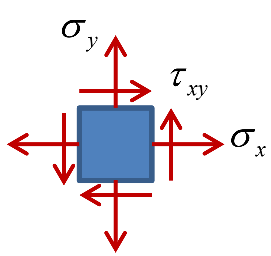
$$\tau'_{xy} = \frac{(\sigma_y - \sigma_x)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

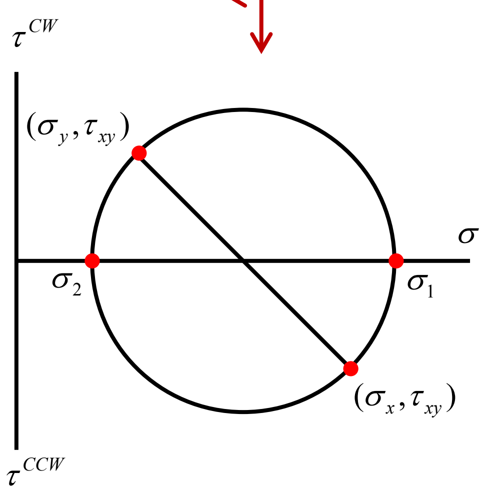
Principal stresses

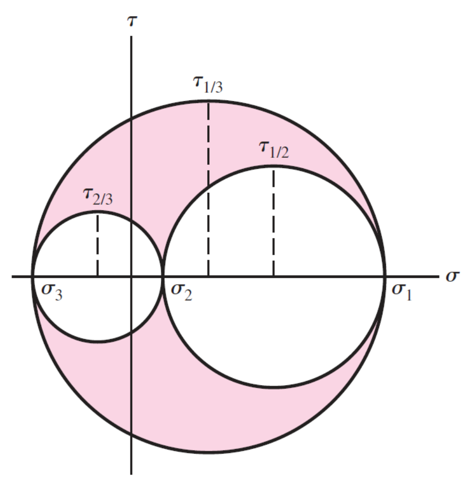
2 Dimension

Principal stress $$\sigma_1, \sigma_2 = \frac{(\sigma_x + \sigma_y)}{2} \pm \left(\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2 \right)^{1/2}$$

Maximum shear stress $$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2}$$







3 Dimension

Principal stress $\sigma_1, \sigma_2, \sigma_3 \qquad \sigma_1 \geq \sigma_2 \geq \sigma_3$

Principal stresses are the solution of the following equation

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$

$$I_2 = \begin{vmatrix} \sigma_y & \tau_{yz} \\ \tau_{zy} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xz} \\ \tau_{zx} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{vmatrix}$$

$$I_3 = \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{vmatrix}$$

Maximum shear stress

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2}$$

Failure Theories

- For simple load, failure can be known by simple test (tension test, compression test).
- For the combination of loading modes, failure theory is required to predict the failure.
- There is no universal theory of failure for the general case of material properties and stress state.
- Consideration are separated depended on metal behavior (ductile or brittle).
- Data used in the failure theories are based on the simple test (tension test, Compression test).

Ductile Materials

$\varepsilon_f \geq 0.05$ (Elongation $\geq$ 5%)

- Maximum shear stress theory (MSS)
- Distortion energy theory (DE)
- Ductile Coulomb-Mohr (DCM)

Brittle Materials

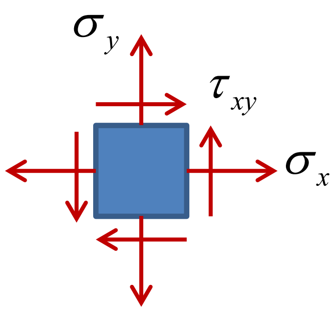
$\varepsilon_f < 0.05$ (Elongation $<$ 5%)

- Maximum normal stress theory (MNS)
- Brittle Coulomb-Mohr (BCM)
- Modifier Mohr (MM)

Maximum shear stress theory (1)

- The **maximum shear stress (MSS) theory** predicts that yielding begins whenever the maximum shear stress in any element equals or exceeds the maximum shear stress in a tension-test specimen of the same material when that specimen begins to yield.
- MSS theory is also referred to as the **Tresca** or **Guest theory**.

Stress state at a point

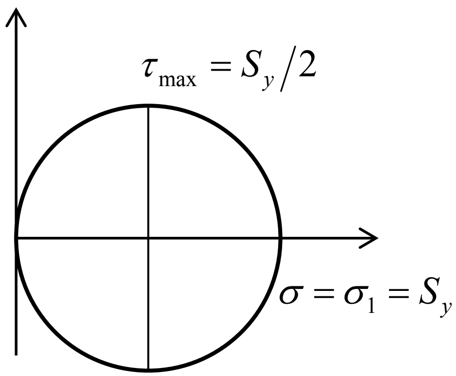


$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2}$$

Tension-test

$$\sigma = \frac{P}{A}$$

$$\tau_{\max} = \frac{\sigma}{2} = \frac{S_y}{2}$$



Yielding begins $\tau_{\max} = \dfrac{\sigma_1 - \sigma_3}{2} \geq \dfrac{S_y}{2}$ or $\sigma_1 - \sigma_3 \geq S_y$ and $S_{sy} = 0.5 S_y$

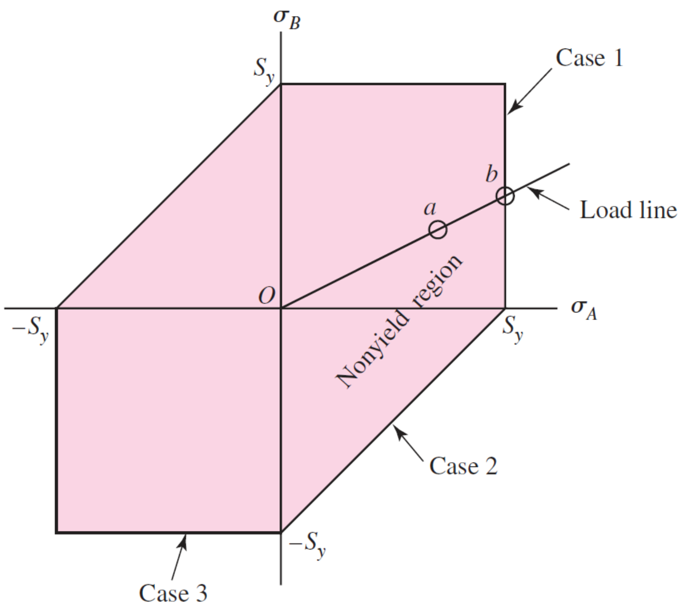
Incorporate a factor of safety $\tau_{\max} = \dfrac{\sigma_1 - \sigma_3}{2} \geq \dfrac{S_y}{2n}$ or $\sigma_1 - \sigma_3 \geq S_y / n$

Maximum shear stress theory (2)

2 Dimension - Plane stress

- Consider at principal direction
- Assuming that principal stress $\sigma_A \geq \sigma_B$
- No stress in the normal plane, hence the other principal stress = 0

σ_A	σ_B	σ_1	σ_3	Yield condition
+	+	σ_A	0	$\sigma_A \geq S_y$
+	-	σ_A	σ_B	$\sigma_A - \sigma_B \geq S_y$
-	-	0	σ_B	$\sigma_B \leq -S_y$



Yield if a stress state is outside the nonyield region

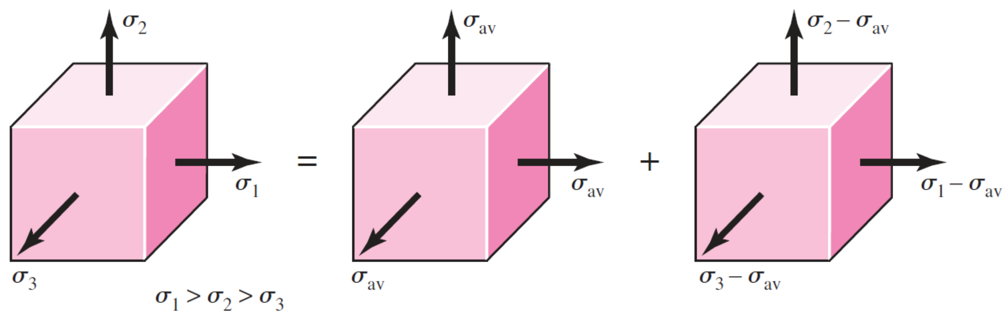
Distortion-Energy theory (1)

- The **distortion energy theory** predicts that yielding occurs when the distortion strain energy per unit volume reaches or exceeds the distortion strain energy per unit volume for yield in simple tension or compression of the same material.
- The distortion energy theory is also called the **von Mises or von Mises-Hencky theory or the octahedral-shear-stress theory**

Angular distortion element

At principal direction

$$\sigma_{av} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$



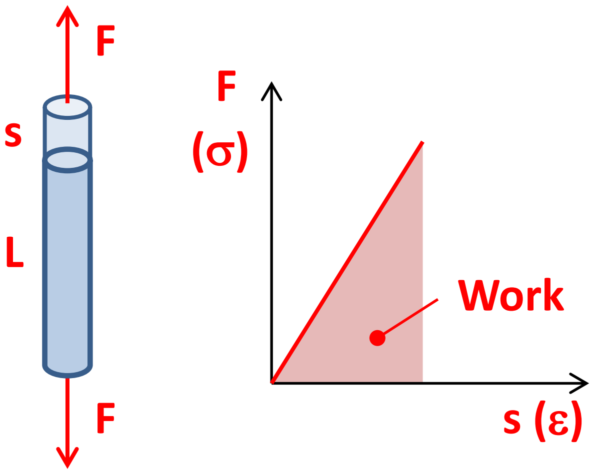
(a) Triaxial stresses (b) Hydrostatic component (c) Distortional component

Pure volume change | Pure angular distortion

Strain energy per unit volume = Strain energy for producing only volume change + Distortion energy

Distortion-Energy theory (2)

Strain energy per unit volume

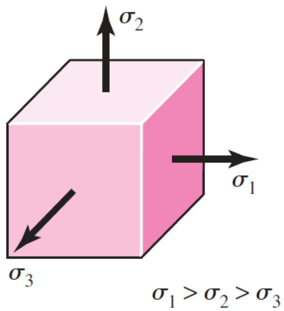


$$\text{work} = \int F \cdot ds = \text{Area under F - S graph}$$

$$\frac{\text{work}}{V} = \int \frac{F}{A} \cdot \frac{ds}{L} = \int \sigma \cdot d\varepsilon = \text{Area under } \sigma \text{ - } \varepsilon \text{ graph}$$

simple tension:

strain energy per unit volume $u = \int \sigma \cdot d\varepsilon = \frac{1}{2}\varepsilon\sigma$



(*a*) Triaxial stresses

$$u = \frac{1}{2}\left[\varepsilon_1\sigma_1 + \varepsilon_2\sigma_2 + \varepsilon_3\sigma_3\right]$$

$$u = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right]$$

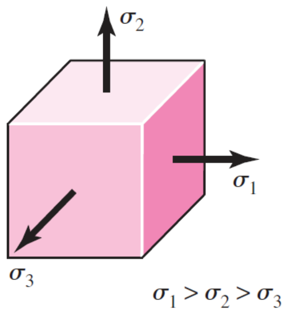
Hooke's law

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_z + \sigma_x)\right]$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right]$$

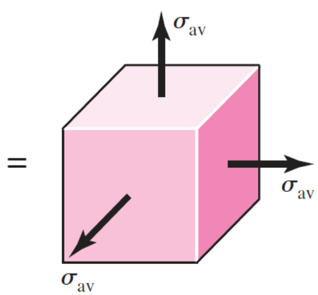
Distortion-Energy theory (3)



(a) Triaxial stresses

strain energy per unit volume

$$u = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right]$$

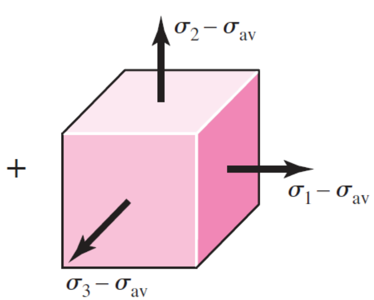


(b) Hydrostatic component

strain energy per unit volume

$$u_v = \frac{3\sigma_{av}^2}{2E}(1-2\nu)$$

$$u_v = \frac{1-2\nu}{2E}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\sigma_1\sigma_2 + 2\sigma_2\sigma_3 + 2\sigma_3\sigma_1)$$



(c) Distortional component

Distortion energy

$$u_d = u - u_v = \frac{1+\nu}{3E}\left[\frac{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2}{2}\right]$$

Distortion-Energy theory (4)

- The **distortion energy theory** predicts that yielding occurs when the distortion strain energy per unit volume reaches or exceeds the distortion strain energy per unit volume for yield in simple tension or compression of the same material.

Distortion strain energy at a point

$$u_d = \frac{1+\nu}{3E}\left[\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}\right]$$

Tension-test

At yield $\quad \sigma_1 = S_y \quad \sigma_2 = \sigma_3 = 0$

$$u_d = \frac{1+\nu}{3E}S_y^2$$

Yielding begins when

$$\frac{1+\nu}{3E}\left[\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}\right] \ge \frac{1+\nu}{3E}S_y^2$$

$$\left[\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}\right]^{1/2} \ge S_y$$

Von Mises stress

Distortion-Energy theory (5)

Von Mises stress

$$\sigma' = \left[\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}\right]^{1/2}$$

In xyz coordinate, the von Mises stress can be calculated from

$$\sigma' = \frac{1}{\sqrt{2}}\left[(\sigma_x-\sigma_y)^2+(\sigma_y-\sigma_z)^2+(\sigma_z-\sigma_x)^2+6(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2)\right]^{1/2}$$

Yielding begins when

Von Mises stress σ' $\geq$ Yield strength S_y

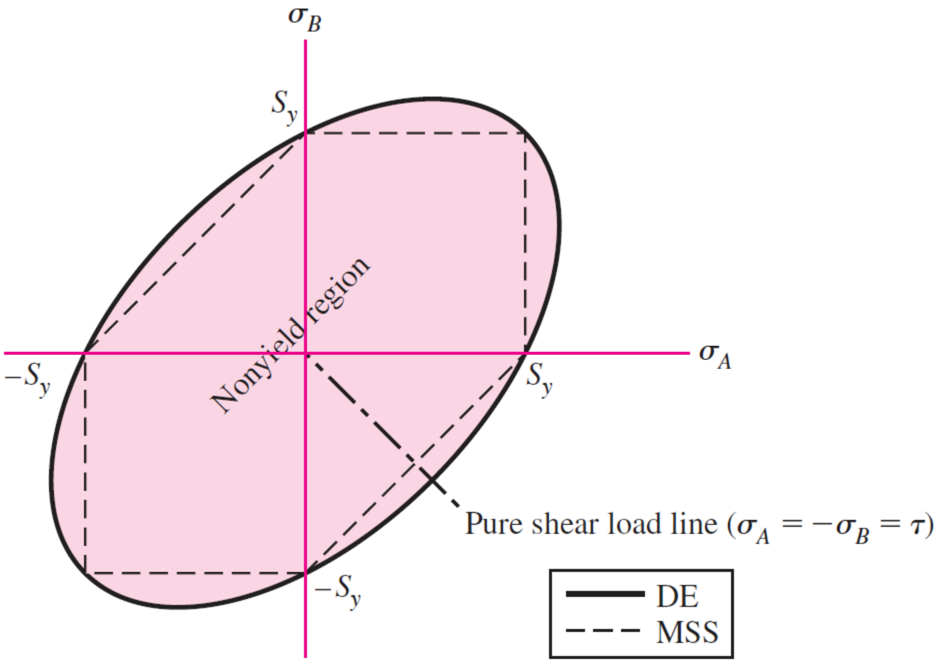
Incorporate a factor of safety $\sigma' \geq \dfrac{S_y}{n}$

Distortion-Energy theory (6)

2D - Plane stress

- Consider at principal direction
- Assuming that principal stress $\sigma_A \geq \sigma_B$
- No stress in the normal plane, hence the other principal stress = 0

Von Mises stress $\sigma' = (\sigma_A^2 - \sigma_A \sigma_B + \sigma_B^2)^{1/2}$

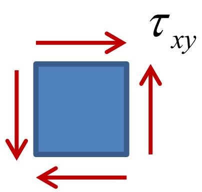


Yield if a stress state is outside the nonyield region

- The nonyield region of the distortion energy theory is wider than the region of the Maximum shear stress theory.
- The prediction from the distortion energy agrees well with all data for ductile behavior. Hence, it is the most widely used theory for ductile materials and is recommended for design problems.

Distortion-Energy theory (7)

2 Dimension - Plane stress + pure shear



$$\sigma_x = \sigma_y = 0$$

von Mises stress equation

$$\sigma' = \frac{1}{\sqrt{2}}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)\right]^{1/2}$$

$$\sigma' = (3\tau_{xy}^2)^{1/2}$$

Yielding begins when

$$\sigma' = (3\tau_{xy}^2)^{1/2} \geq S_y$$

$$\tau_{xy} \geq \frac{S_y}{\sqrt{3}} = 0.577 S_y$$

Shear yield strength

$$S_{sy} = 0.577 S_y$$

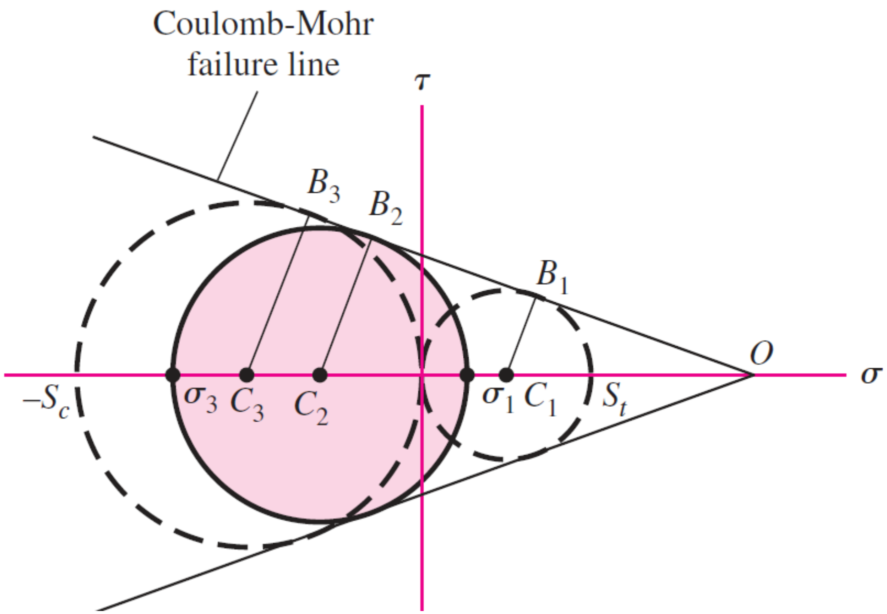
Example

A material has the yield strength $S_{yc} = S_{yt} = 100$ MPa, and $\varepsilon_f = 0.55$. Determine the factor of safety of the following cases.

(Mpa)	σ_x	σ_y	τ_{xy}
a	70	70	0
b	60	40	-15
c	0	40	45
d	-40	-60	15
e	30	30	30

Coulomb-Mohr Theory (Ductile Materials) (1)

- **Can be used for materials whose strengths in tension and compression are not equal.**
- **Use data from tension test and compression test to draw Mohr's circles**
- **Draw failure enveloped tangent to the circles**
- **Yield if a stress state is outside the envelope**



Triangles OB_iC_i are similar, therefore

$$\frac{B_2C_2 - B_1C_1}{C_1C_2} = \frac{B_3C_3 - B_1C_1}{C_1C_3}$$

$$B_1C_1 = S_t/2 \qquad \text{origin} - C_1 = S_t/2$$

$$B_2C_2 = (\sigma_1 - \sigma_3)/2 \qquad \text{origin} - C_2 = (\sigma_1 + \sigma_3)/2$$

$$B_3C_3 = S_c/2 \qquad \text{origin} - C_3 = S_c/2$$

$$\frac{\sigma_1}{S_t} - \frac{\sigma_3}{S_c} = 1$$

Yielding begins $\quad \dfrac{\sigma_1}{S_t} - \dfrac{\sigma_3}{S_c} \geq 1$

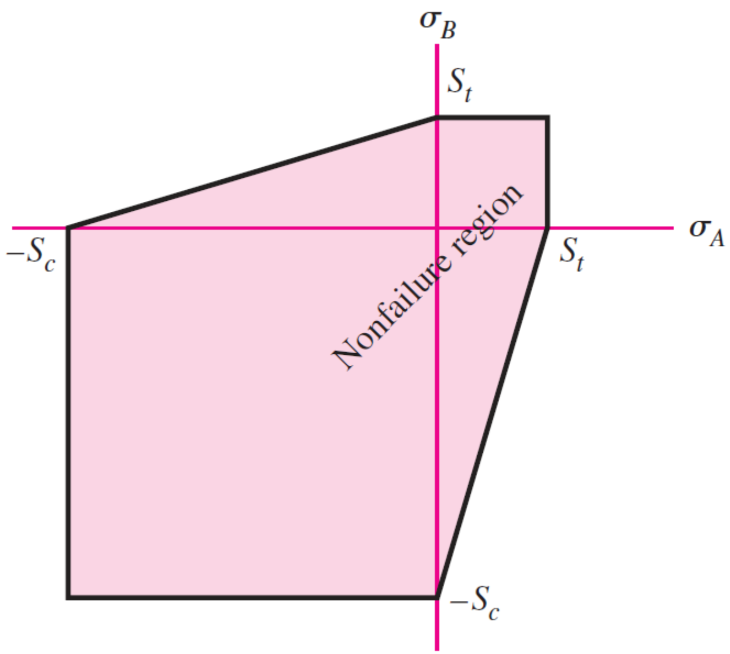
Incorporate a factor of safety $\quad \dfrac{\sigma_1}{S_t} - \dfrac{\sigma_3}{S_c} \geq \dfrac{1}{n}$

Coulomb-Mohr Theory (Ductile Materials) (2)

2D - Plane stress

- Consider at principal direction
- Assuming that principal stress $\sigma_A \geq \sigma_B$
- No stress in the normal plane, hence the other principal stress = 0

σ_A	σ_B	σ_1	σ_3	Yield condition
+	+	σ_A	0	$\sigma_A \geq S_t$
+	-	σ_A	σ_B	$\dfrac{\sigma_A}{S_t} - \dfrac{\sigma_B}{S_c} \geq 1$
-	-	0	σ_B	$\sigma_B \leq -S_c$



Yield if a stress state is outside the nonyield region

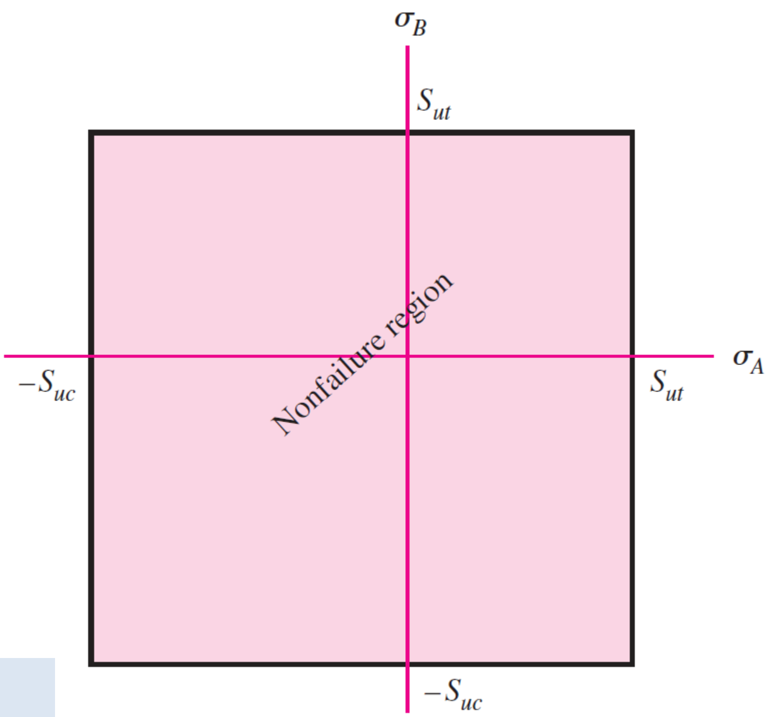
Maximum-Normal-Stress Theory (Brittle)

- **The maximum normal stress (MNS) theory states that failure occurs whenever one of the three principal stresses equals or exceeds the strength.**

Principal stress $\sigma_1 \geq \sigma_2 \geq \sigma_3$

Yielding begins $\sigma_1 \geq S_{ut}$ or $\sigma_3 \leq -S_{uc}$

Incorporate a factor of safety $\sigma_1 \geq \dfrac{S_{ut}}{n}$ or $\sigma_3 \leq -\dfrac{S_{uc}}{n}$



Note

Yield strength of the brittle materials can not be observed, hence the ultimate tensile strength or ultimate compressive strength are used instead

Modifications of the Mohr Theory (Brittle)

Brittle-Coulomb-Mohr

Plane stress + factor of safety

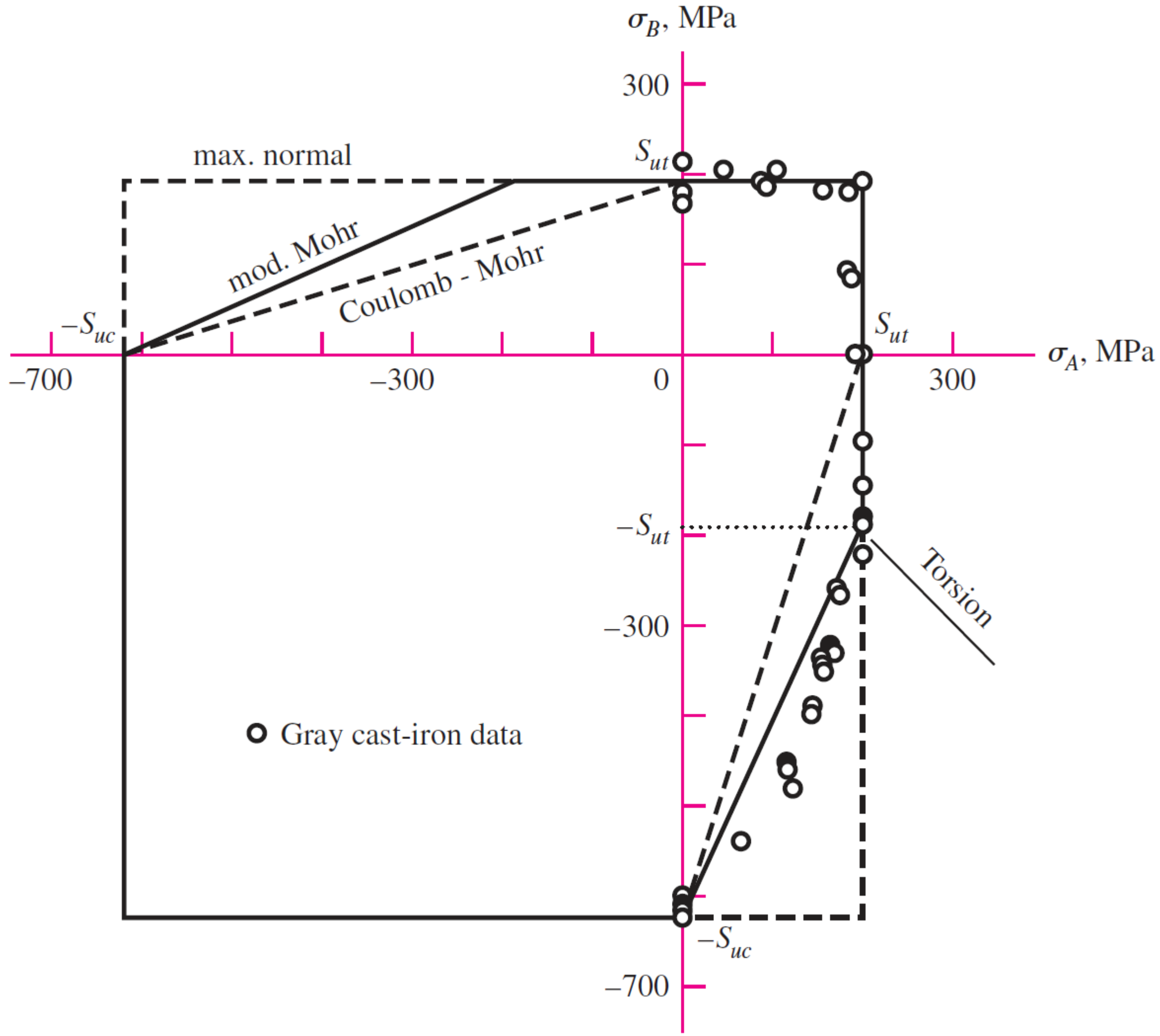
σ_A	σ_B	σ_1	σ_3	Yield condition
+	+	σ_A	0	$\sigma_A \geq S_{ut}/n$
+	-	σ_A	σ_B	$\frac{\sigma_A}{S_{ut}} - \frac{\sigma_B}{S_{uc}} \geq \frac{1}{n}$
-	-	0	σ_B	$\sigma_B \leq -S_{uc}/n$

Modified Mohr

Plane stress + factor of safety

σ_A	σ_B	σ_1	σ_3	Yield condition
+	+	σ_A	0	$\sigma_A \geq S_{ut}/n$
+	-	σ_A $\left\lvert \sigma_B/\sigma_A \right\rvert \leq 1$	σ_B	$\sigma_A \geq S_{ut}/n$
+	-	σ_A $\left\lvert \sigma_B/\sigma_A \right\rvert > 1$	σ_B	$\frac{(S_{uc}-S_{ut})\sigma_A}{S_{uc}S_{ut}} - \frac{\sigma_B}{S_{uc}} \geq \frac{1}{n}$
-	-	0	σ_B	$\sigma_B \leq -S_{uc}/n$

Modifications of the Mohr Theory (Brittle)



Selection of Failure Criteria