# Small Areas, Benchmarking, and Political Battles: Today's Novel Demands in Small-Area Estimation

Rebecca C. Steorts

Department of Statistics
Carnegie Mellon University
joint with Malay Ghosh, Gauri Datta, and Jerry Maples

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*Small area estimation is about disaggregating surveys to small noisy subgroups.*

An area $i$ is small if the sample size is not large enough to support direct estimates $\hat{\theta}_i$ of adequate precision.

- An "area" could be geographic, demographic, etc.
- Borrow strength from related areas.
- Hierarchical and Empirical Bayes methods.

*Many applications have multiple levels of resolution that call for aggregating estimates.*

- Model-based estimates for small areas often do not aggregate to the direct estimates for larger areas.
- Having model-based estimates that *do* aggregate properly is often a political necessity.

**Benchmarking**

*Benchmarking* is adjusting model-based estimates such that they aggregate to direct estimates for larger areas.

Helps deal with possible model misspecification and overshrinkage.

*Goals: Develop general class of benchmarked Bayes estimators and explore effects on the MSE.*

In Datta et al. (2011), we extend Wang et al. (2008), developing a general class of benchmarked Bayes estimators.

- No distributional assumptions.
- Linear or nonlinear estimators.
- Benchmark the weighted mean and/or weighted variability.
- Multivariate version.
- Includes many previously proposed estimators as special cases.

## Objective

Minimizing a posterior risk

$$\min_{\boldsymbol{\delta}} \sum_{i=1}^{m} \phi_i E[(\delta_i - \theta_i)^2 | \hat{\boldsymbol{\theta}}]$$

subject to the benchmarking constraint(s)

$$\sum_{i=1}^{m} w_i \delta_i = t \text{ and possibly } \sum_{i=1}^{m} w_i (\delta_i - t)^2 = h \ .$$

- Derive the benchmarked Bayes estimators $\hat{\boldsymbol{\theta}}^{BM}$ in closed form.
- $\hat{\boldsymbol{\theta}}^{BM}$ = Bayes estimator $\hat{\boldsymbol{\theta}}^{B}$ plus a correction factor.

*How does benchmarking affect the errors of the estimates?*

Using Fay-Herriot model and standard benchmarking constraint:

- Theoretically compare $MSE[\hat{\theta}^{EB}]$ and $MSE[\hat{\theta}^{EBM}]$.
  - Builds off Prasad and Rao (1990) and Wang et al. (2008); Ugarte et al. (2009).
- Derive two estimators of $MSE[\hat{\theta}^{EBM}]$ (asymptotically unbiased and parametric bootstrap).
- Evaluate methods using Small Area Income and Poverty Estimate Program (U.S. Census Bureau).

[Steorts and Ghosh (2013)]

*With m small areas, the increase in MSE due to benchmarking is* $O(m^{-1})$.

*This is shown via a second-order asymptotic expansion.*

Consider the area-level effects model of Fay and Herriot (1979):

$$\hat{\theta}_i|\theta_i \overset{ind}{\sim} N(\theta_i, D_i)$$
$$\theta_i|\boldsymbol{\beta}, \sigma_u^2 \overset{ind}{\sim} N(x_i'\boldsymbol{\beta}, \sigma_u^2), \qquad i = 1, \ldots, m.$$

Assume $D_i$ is known and $\sigma_u^2$ and $\boldsymbol{\beta}$ are unknown.

- Estimate $\sigma_u^2$ by moment estimator $\tilde{\sigma}_u^2$. Then $\hat{\sigma}_u^2 = \max\{\tilde{\sigma}_u^2, 0\}$.
- Estimate $\boldsymbol{\beta}$ by a GLS-type estimator.
- Derive the benchmarked empirical Bayes estimator $\hat{\theta}^{EBM}$.

### Theorem

$MSE[\hat{\theta}_i^{EBM}] = g_{1i}(\sigma_u^2) + g_{2i}(\sigma_u^2) + g_{3i}(\sigma_u^2) + g_4(\sigma_u^2) + o(m^{-1}),$
where

$$g_{1i}(\sigma_u^2) = \frac{D_i \sigma_u^2}{D_i + \sigma_u^2} = O(1),$$

$$g_{2i}(\sigma_u^2) \approx \text{diagonal of hat matrix } h_{ii}^V = O(m^{-1}),$$

$$g_{3i}(\sigma_u^2) \approx \text{noise in estimating } \sigma_u^2 = O(m^{-1}),$$

$$g_4(\sigma_u^2) \approx \text{avg. variance specific to each } \hat{\theta}_i = O(m^{-1}).$$

- Note: $MSE[\hat{\theta}_i^{EB}] = g_{1i}(\sigma_u^2) + g_{2i}(\sigma_u^2) + g_{3i}(\sigma_u^2) + o(m^{-1})$.
- The difference in MSEs is $g_4(\sigma_u^2)$.

We extend the method of Butar and Lahiri (2003) to derive a parametric bootstrap estimator $V_i^{\text{B-BOOT}}$ of $MSE[\hat{\theta}_i^{EBM}]$.

- Use parametric bootstrapping from Fay-Herriot model to correct plug-in estimates of $g_{1i}(\sigma_u^2), g_{2i}(\sigma_u^2)$, and $g_4(\sigma_u^2)$.
- Use the same bootstrap to estimate $g_{3i}(\sigma_u^2)$ directly.
- Combination is asymptotically unbiased:

$$E[V_i^{\text{B-BOOT}}] = MSE[\hat{\theta}_i^{\text{EBM}}] + o(m^{-1}).$$

*How does benchmarking perform in applications?*

- Small Area Income and Poverty Estimates (SAIPE) program (U.S. Census Bureau): model-based estimates of the number of poor children (aged 5–17).
- Model-based state estimates were benchmarked to a direct estimate of national child poverty by raking.
- Direct estimates came from from the Annual Social and Economic (ASEC) Supplement of the Current Population Survey (CPS) and the American Community Survey (ACS).
- Weights $w_i \propto$ estimated number of children in each state.

Recall the model of Fay and Herriot (1979):

$$\hat{\theta}_i|\theta_i \overset{ind}{\sim} N(\theta_i, D_i)$$
$$\theta_i|\beta, \sigma_u^2 \overset{ind}{\sim} N(x_i'\boldsymbol{\beta}, \sigma_u^2), \qquad i = 1, \ldots, m$$

- where $D_i > 0$ are known,
- $\theta_i$ are the true state level poverty rates,
- $\hat{\theta}_i$ are the direct state estimates.

Employ EB on unknown $\boldsymbol{\beta}$ and $\sigma_u^2$.

- We consider data from 1997 and 2000.
- The data from 2000 behaves as our theory indicates: $\text{MSE}[\hat{\theta}^{EBM}]$ are slightly larger than $\text{MSE}[\hat{\theta}^{EB}]$.
- The same is true when we bootstrap.

Table: Table of estimates for 1997

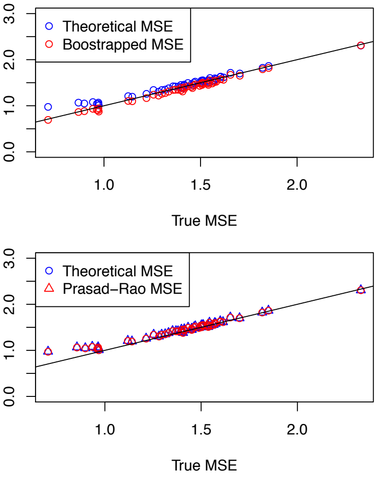
| | Estimates | | | MSEs | | | Bootstrap | |
|---|---|---|---|---|---|---|---|---|
| $i$ | $\hat{\theta}_i$ | $\hat{\theta}_i^{EB}$ | $\hat{\theta}_i^{EBM1}$ | $\hat{\theta}_i$ | $\hat{\theta}_i^{EB}$ | $\hat{\theta}_i^{EBM1}$ | $\hat{\theta}_i^{EB}$ | $\hat{\theta}_i^{EBM1}$ |
| 12 | 18.98 | 13.72 | 13.89 | 20.87 | 2.45 | 2.48 | 1.24 | 1.26 |
| 13 | 17.56 | 13.64 | 13.82 | 12.38 | 1.70 | 1.73 | 0.23 | 0.25 |
| 14 | 14.57 | 15.72 | 15.89 | 3.56 | 3.45 | 3.47 | −0.06 | −0.05 |
| 15 | 11.07 | 12.53 | 12.70 | 7.58 | 1.84 | 1.86 | −0.23 | −0.22 |
| 16 | 11.09 | 11.21 | 11.38 | 8.49 | 1.74 | 1.76 | −0.24 | −0.22 |
| 17 | 11.01 | 13.48 | 13.65 | 9.34 | 1.61 | 1.63 | −0.15 | −0.14 |
| 18 | 23.12 | 20.78 | 20.95 | 13.98 | 1.37 | 1.40 | −0.12 | −0.11 |
| 19 | 21.08 | 24.15 | 24.32 | 15.19 | 1.80 | 1.82 | 0.40 | 0.42 |
| 20 | 13.18 | 12.44 | 12.61 | 13.63 | 2.09 | 2.11 | 0.56 | 0.57 |
| 21 | 9.90 | 13.16 | 13.33 | 9.28 | 1.65 | 1.67 | −0.03 | −0.01 |
| 22 | 19.66 | 14.38 | 14.56 | 7.66 | 2.46 | 2.48 | 1.02 | 1.04 |
| 23 | 13.78 | 16.86 | 17.03 | 4.04 | 3.11 | 3.13 | 0.38 | 0.39 |

- Strange behavior for 1997; problem occurs when $\hat{\sigma}_u^2$ is 0.
- Note that

$$V_i^{\text{B-BOOT}} = g_{1i}(\hat{\sigma}_u^2) + \{g_{1i}(\hat{\sigma}_u^2) - E_*[g_{1i}(\hat{\sigma}_u^{*2})]\} + O(m^{-1}).$$

- $g_{1i}(\hat{\sigma}_u^2) = D_i\hat{\sigma}_u^2(D_i + \hat{\sigma}_u^2)^{-1} = O(1)$.
  - For 1997 dataset this term is 0.
  - This causes many of the bootstrap estimates of the MSE of the benchmarked estimators to be negative.
- Theoretical (asymptotic) MSE escapes problem since

$$P(\tilde{\sigma}_u^2 \leq 0) = O(m^{-r}) \; \forall \; r > 0.$$




Simulation study for 1997

- Unified framework for one-stage benchmarking.
- The increase in MSE due to benchmarking is negligible.
- Derived two estimators of our MSE (asymptotically unbiased and parametric bootstrap).
- Recommend use of estimator of the MSE of the benchmarked EB estimator.
  - Fast calculation.
  - Parametric bootstrap yields undesirable results.

- Spatial and temporal smoothing for SAE and benchmarking.
- Application to high dimensional dataset (both in covariates and parameter space) and more standard applications in SAE.
- Comparing to frequentists benchmarks under MSE comparisons (under bootstrapping).
- Validations under CV and model-checking.

Questions: beka@cmu.edu

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We benchmark a weighted mean or both a weighted mean and variability.

- $\hat{\theta}_1, \ldots, \hat{\theta}_m =$ direct estimators of the $m$ small area means $\theta_1, \ldots, \theta_m$.
- Find the benchmarked Bayes estimator

$$\hat{\boldsymbol{\theta}}^{BM1} = (\hat{\theta}_1^{BM1}, \ldots, \hat{\theta}_m^{BM1})$$

  of $\boldsymbol{\theta}$ such that $\sum_{i=1}^m w_i \hat{\theta}_i^{BM1} = t$, where $t$ is prespecified from some other source or $t = \sum_{i=1}^m w_i \hat{\theta}_i$.
- The $w_i$ are known weights, where $\sum_{i=1}^m w_i = 1$.

- Goal:

$$\min_{\boldsymbol{\delta}} \sum_{i=1}^{m} \phi_i E[(\delta_i - \theta_i)^2 | \hat{\boldsymbol{\theta}}]$$

  such that the $\delta_i$'s satisfy $\bar{\delta}_w = \sum_{i=1}^m w_i \delta_i = t$.
- $\hat{\theta}_i^B$ = posterior mean of $\theta_i$ under a particular prior.
- $\bar{\hat{\theta}}_w^B = \sum_{i=1}^m w_i \hat{\theta}_i^B$.
- $r = (r_1, \ldots, r_m)'$ where $r_i = w_i/\phi_i$, and define $s = \sum_{i=1}^m w_i^2/\phi_i$.

## Theorem 1

$$\hat{\boldsymbol{\theta}}^{BM1} = \hat{\boldsymbol{\theta}}^{B} + s^{-1}(t - \bar{\hat{\theta}}_w^B)r.$$

minimizes $\sum_{i=1}^{m} \phi_i E[(\delta_i - \theta_i)^2 | \hat{\boldsymbol{\theta}}]$ subject to $\bar{\delta}_w = t$.

(The theorem extends to a multivariate setting)

- We can also benchmark using (i) $\sum_i w_i \hat{\theta}_i^{BM2} = t$ and (ii) $\sum_i w_i (\hat{\theta}_i^{BM2} - t)^2 = H$, where H is defined below. Maybe we think our estimates are too close together, for example.
- This can be extended to a multivariate setting.

## Theorem 2

Subject to (i) and (ii), the benchmarked Bayes estimators of $\theta_i$ are given by

$$\hat{\theta}_i^{BM2} = \hat{\theta}_i^B + (t - \bar{\hat{\theta}}_w^B) + (a_{CB} - 1)(\hat{\theta}_i^B - \bar{\hat{\theta}}_w^B),$$

where $a_{CB} = H / \sum_{i=1}^m w_i (\hat{\theta}_i^B - \bar{\hat{\theta}}_w^B)^2$. Note that $a_{CB} \geq 1$ when $H = \sum_{i=1}^m w_i E[(\theta_i - \bar{\theta}_w)^2 | \hat{\boldsymbol{\theta}}]$.

Consider the area-level effects model of Fay and Herriot (1979):

$$\hat{\theta}_i|\theta_i \overset{ind}{\sim} N(\theta_i, D_i)$$
$$\theta_i|\beta, \sigma_u^2 \overset{ind}{\sim} N(x_i'\boldsymbol{\beta}, \sigma_u^2), \qquad i = 1, \ldots, m$$

Assume $D_i$ is known and $\sigma_u^2$ and $\beta$ are unknown.

- Estimate $\sigma_u^2$ by moment estimator $\tilde{\sigma}_u^2$. Then $\hat{\sigma}_u^2 = \max\{\tilde{\sigma}_u^2, 0\}$.
- We estimate $\boldsymbol{\beta}$ by $\tilde{\boldsymbol{\beta}} = (X'V^{-1}X)^{-1}X'V^{-1}\hat{\boldsymbol{\theta}}$, where $V = \text{Diag}\{\sigma_u^2 + D_1, \ldots, \sigma_u^2 + D_m\}$.
- Benchmarked empirical Bayes estimator derived by Datta et al. (2011) is $\hat{\theta}^{EBM1} = \hat{\theta}_i^{EB} + (\bar{\hat{\theta}}_w - \bar{\hat{\theta}}_w^{EB})$.
- $\hat{\theta}_i^{EB} = (1 - \hat{B}_i)\hat{\theta}_i + \hat{B}_i x_i'\tilde{\boldsymbol{\beta}}(\hat{\sigma}_u^2)$, where $\hat{B}_i = D_i(\hat{\sigma}_u^2 + D_i)^{-1}$.

Define $h_{ij}^V = x_i'(X'V^{-1}X)^{-1}x_j$. Under some mild regularity conditions, we can find a second-order approximation of the MSE of the benchmarked empirical Bayes estimator.

## Theorem 4

$E[(\hat{\theta}_i^{EBM1} - \theta_i)^2] = g_{1i}(\sigma_u^2) + g_{2i}(\sigma_u^2) + g_{3i}(\sigma_u^2) + g_4(\sigma_u^2) + o(m^{-1})$, where

$$g_{1i}(\sigma_u^2) = B_i\sigma_u^2, \quad g_{2i}(\sigma_u^2) = B_i^2 h_{ii}^V,$$

$$g_{3i}(\sigma_u^2) = B_i^3 \text{Var}(\tilde{\sigma}_u^2),$$

$$g_4(\sigma_u^2) = \sum_{i=1}^m w_i^2 B_i^2 V_i - \sum_{i=1}^m \sum_{j=1}^m w_i w_j B_i B_j h_{ij}^V, \text{ and}$$

$$\text{Var}(\tilde{\sigma}_u^2) = 2(m-p)^{-2} \sum_{k=1}^m (\sigma_u^2 + D_k)^2 + o(m^{-1}).$$

We use the methods of Butar and Lahiri (2003) and use the following bootstrap model:

$$\hat{\theta}_i^* | u_i^* \overset{ind}{\sim} N(x_i'\beta + u_i^*, D_i)$$
$$u_i^* \overset{ind}{\sim} N(0, \hat{\sigma}_u^2).$$

We use the parametric bootstrap twice. We first use it to estimate $g_{1i}(\sigma_u^2), g_{2i}(\sigma_u^2)$, and $g_4(\sigma_u^2)$. We then use it to estimate $E[(\hat{\theta}_i^{EB} - \hat{\theta}_i^{B})^2] = g_{3i}(\sigma_u^2) + o(m^{-1})$.

Our proposed estimate of $MSE[\hat{\theta}_i^{EBM1}]$ is

$$\begin{aligned} V_i^{\text{B-BOOT}} &= 2[g_{1i}(\hat{\sigma}_u^2) + g_{2i}(\hat{\sigma}_u^2) + g_4(\hat{\sigma}_u^2)] \\ &- E_* \left\{ g_{1i}(\hat{\sigma}_u^{*2}) + g_{2i}(\hat{\sigma}_u^{*2}) + g_4(\hat{\sigma}_u^{*2}) \right\} \\ &+ E_*[(\hat{\theta}_i^{EB*} - \hat{\theta}_i^{EB})^2]. \end{aligned}$$

- Our estimate $\hat{\sigma}_u^{*2}$ is the estimate of $\sigma_u^2$ that is calculated using the $\hat{\theta}_i^*$ values.
- Note that $\hat{\theta}_i^{EB*}$ is calculated using $\hat{\sigma}_u^{*2}$ and $\hat{\theta}_i$ (not $\hat{\theta}_i^*$).

We extend the methodology of Butar and Lahiri (2003) to find a parametric bootstrap estimator of the MSE of the benchmarked EB estimator. Then we can show

**Theorem 6**

$$E[V_i^{\text{B-BOOT}}] = MSE[\hat{\theta}_i^{\text{EBM1}}] + o(m^{-1}).$$