HOMEWORK 1

1. Find the number of ways of arranging the 26 letters in the English alphabet in a row such that there are exactly 5 letters between x and y.

2. Find the number of odd integers between 3000 and 8000 in which no digit is repeated.

3. Evaluate $1\cdot 1! + 2\cdot 2! + 3\cdot 3! + \cdots + n\cdot n!$ where $n\in\mathbb{N}$.

4. Show that for any $n\in\mathbb{N}$, the number of positive divisors of n^2 is always odd.

5. Find the number of $(m+n)-$digit binary sequences with m 0's and n 1's such that no two 1's are adjacent, where $n\le m+1$.

6. A box contains 7 identical white balls and 5 identical black balls. They are to be drawn randomly, one at time without replacement, until the box is empty. Find the probability that the 6^{th} ball drawn is white, while before that exactly 3 black balls are drawn.

7. Let $X=\{1,2,3,\ldots,1000\}$. Find the number of 2-element subsets $\{a,b\}$ of X such that the product $a\cdot b$ is divisible by 5.

8. Let $k,n\in\mathbb{N}$. Show that the number of ways to seat kn people around k distinct tables such that there are n people in each table is given by $\dfrac{(kn)!}{n^k}$.

9. Using the numbers 1,2,3,4,5, we can form $5!(=120)$ 5-digit numbers in which the 5 digits are all distinct. If these numbers are listed in increasing order:

$$\underset{1st}{12345},\ \underset{2nd}{12354},\ \underset{3rd}{12435},\ \ldots,\ \underset{120the}{54321},$$

(i) find the position of the number 35421;

(ii) find the 100^{th} number in the list.

10. Let $r,b\in\mathbb{N}$ with $r\le n$. A permutation $x_1x_2\cdots x_{2n}$ of the set $\{1,2,3,\ldots,2n\}$ is said to have property $P(r)$ if $|x_i - x_{i+1}| = r$ for at least one i in $\{1,2,\ldots,2n-1\}$. Show that, for each n and r, there are more permutations with property $P(r)$ than without.

11. Prove by a combinatorial argument that each of the following numbers is always an integer for each $n \in \mathbb{N}$:

(i) $\dfrac{(6n)!}{5^n 3^{2n} 2^{4n}}$, (ii) $\dfrac{(n!)!}{(n!)^{(n-1)!}}$.

12. Let $r, n, k \in \mathbb{N}$ such that $r \geq nk$. Find the number of ways of distributing r identical objects into n distinct boxes so that each box holds at least k objects.

13. Given $r, n \in \mathbb{N}$ with $r \geq n$, let $L(r,n)$ denote the number of ways of distributing r distinct objects into n identical boxes so that no box is empty and the objects in each box can be arranged in a row. Find $L(r,n)$ in terms of r and n.

14. Find the number of integer solutions to each of the following equations:

(i) $x_1 + x_2 + x_3 + x_4 = 30$, $2 \leq x \leq 7$ and $x_i \geq 0$ for each $i = 2,3,4$.

(ii) $x_1 + x_2 + x_3 + x_4 \leq 2009$.

(iii) $rx_1 + x_2 + \cdots + x_n = kr$, where, $r, n, k \in \mathbb{N}$.

15. For $n \geq 4$, let $r(n)$ denote the number of interior regions of a convex n-gon divided by all its diagonals if no three diagonals are concurrent within the n-gon. For instance, $r(4) = 4$ and $r(5) = 11$. Prove that $r(n) = \binom{n}{4} + \binom{n-1}{2}$.

16. Let $S = \{1,2,3,\ldots,2552\}$. In each of the following cases, find the number of 3-element subsets $\{a,b,c\}$ of S satisfying the given condition:

(i) $3 \mid (a+b+c)$;

(ii) $4 \mid (a+b+c)$.

17. A set $S = \{a_1, a_2, \ldots, a_r\}$ of positive integers, where $r \in \mathbb{N}$ and $a_1 < a_2 < \cdots < a_r$, is said to be m-saperated ($m \in \mathbb{N}$) if $a_i - a_{i-1} \geq m$, for each $i = 2,3,\ldots,r$. Let $X = \{1,2,\ldots,n\}$. Find the number of r-element subsets of X which are $m-$separated, where $0 \leq r \leq n-(m-1)(r-1)$.